A

COURSE

OF

MATHEMATICS.

IN THREE VOLUMES.

COMPOSED FOR

THE USE OF THE ROYAL MILITARY ACADEMY,

BY ORDER OF HIS LORDSHIP

THE MASTER GENERAL OF THE ORDNANCE.

BY

CHARLES HUTTON, LL.D. F.R.S.

LATE PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY.

THE SIXTH EDITION,

ENLARGED AND CORRECTED.

VOL. II.

LONDON:

PRINTED FOR F. C. AND J. RIVINGTON; G. WILKIE AND J. ROBINSON; J. WALKER; G. ROBINSON; LACKINGTON, ALLEN, AND CO.; VERNOR, HOOD, AND SHARPE; G. KEARSLEY; LONGMAN, HURST, REES, ORME, AND BROWN; CADELL AND DAVIES; J. CUTHELL; B. CROSBY AND CO.; J. RICHARDSON; J. M. RICHARDSON; BLACK, PARRY, AND KINGSBURY; GALE AND CURTIS; AND J. JOHNSON AND CO.

1811.

T. Davison, Lombard-street,
Whitefriars, London.

CONTENTS OF VOL. II.

ERRATA IN VOL. II.

Page 294, line 12, for $(\sqrt{a^2-x^2})$, read $\sqrt{(a^2-x^2)}$.
302, line 24, for $\sqrt{a^2+x^2}$, read $\sqrt{\overline{a^2}+\overline{x^2}}$.
309, line 23, for mn, read $\frac{1}{mn}$.
349, line 19, for tv, read $t\dot{v}$.

A

COURSE

OF

MATHEMATICS, &c.

PLANE TRIGONOMETRY.

DEFINITIONS.

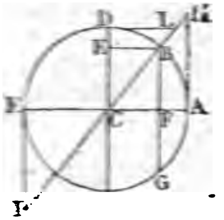
1. PLANE TRIGONOMETRY treats of the relations and calculations of the sides and angles of plane triangles.

2. The circumference of every circle (as before observed in Geom. Def. 56) is supposed to be divided into 360 equal parts, called Degrees; also each degree into 60 Minutes, and each minute into 60 Seconds, and so on. Hence a semicircle contains 180 degrees, and a quadrant 90 degrees.

3. The Measure of an angle (Def. 57, Geom.) is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is estimated by the number of degrees contained in that arc.

Hence, a right angle, being measured by a quadrant, or quarter of the circle, is an angle of 90 degrees; and the sum of the three angles of every triangle, or two right angles, is equal to 180 degrees. Therefore, in a right-angled triangle, taking one of the acute angles from 90 degrees, leaves the other acute angle; and the sum of the two angles, in any triangle, taken from 180 degrees, leaves the third angle; or one angle being taken from 180 degrees, leaves the sum of the other two angles.

4. Degrees are marked at the top of the figure with a small °, minutes with ′, seconds with ″, and so on. Thus, 57° 30′ 12″, denote 57 degrees 30 minutes and 12 seconds.



5. The Complement of an arc, is what it wants of a quadrant or 90°. Thus, if AD be a quadrant, then BD is the complement of the arc AB; and, reciprocally, AB is the complement of BD. So that, if AB be an arc of 50°, then its complement BD will be 40°.

6. The Supplement of an arc, is what it wants of a semicircle, or 180°. Thus, if ADE be a semicircle, then BDE is the supplement of the arc AB; and, reciprocally, AB is the supplement of the arc BDE. So that, if AB be an arc of 50°, then its supplement BDE will be 130°.

7. The Sine, or Right Sine, of an arc, is the line drawn from one extremity of the arc, perpendicular to the diameter which passes through the other extremity. Thus, BF is the sine of the arc AB, or of the supplemental arc BDE. Hence the sine (BF) is half the chord (BG) of the double arc (BAG).

8. The Versed Sine of an arc, is the part of the diameter intercepted between the arc and its sine. So, AF is the versed sine of the arc AB, and EF the versed sine of the arc EDB.

9. The Tangent of an arc, is a line touching the circle in one extremity of that arc, continued from thence to meet a line drawn from the centre through the other extremity; which last line is called the Secant of the same arc. Thus, AH is the tangent, and CH the secant, of the arc AB. Also, EI is the tangent, and CI the secant, of the supplemental arc BDE. And this latter tangent and secant are equal to the former, but are accounted negative, as being drawn in an opposite or contrary direction to the former.

10. The Cosine, Cotangent, and Cosecant, of an arc, are the sine, tangent, and secant of the complement of that arc, the Co being only a contraction of the word complement. Thus, the arcs AB, BD, being the complements of each other, the sine, tangent, or secant of the one of these, is the cosine, cotangent, or cosecant of the other. So, BF, the sine of AB, is the cosine of BD; and BK, the sine of BD, is the cosine of AB: in like manner, AH, the tangent of AB, is the cotangent of BD; and DL, the tangent of DB, is the cotangent of AB: also, CH, the secant of AB, is the cosecant of BD; and CL, the secant of BD, is the cosecant of AB.

Corol.

Corol. Hence several remarkable properties easily follow from these definitions; as,

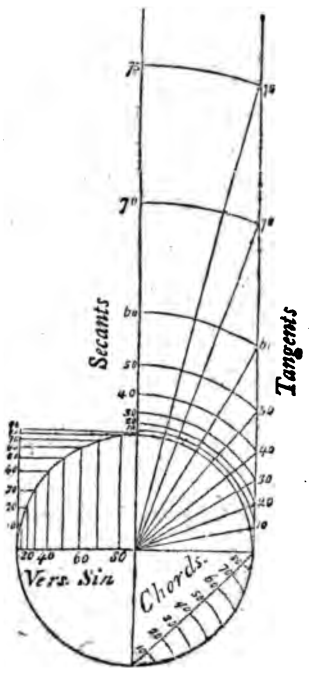
1st, That an arc and its supplement have the same sine, tangent, and secant; but the two latter, the tangent and secant, are accounted negative when the arc is greater than a quadrant or 90 degrees.

2d, When the arc is 0, or nothing, the sine and tangent are nothing, but the secant is then the radius CA, the least it can be. As the arc increases from 0, the sines, tangents, and secants, all proceed increasing, till the arc becomes a whole quadrant AD, and then the sine is the greatest it can be, being the radius CD of the circle; and both the tangent and secant are infinite.

3d, Of any arc AB, the versed sine AF, and cosine BK, or CF, together make up the radius CA of the circle.—The radius CA, the tangent AH, and the secant CH, form a right-angled triangle CAH. So also do the radius, sine, and cosine, form another right-angled triangle CBF or CBK. As also the radius, cotangent, and cosecant, another right-angled triangle CDL. And all these right-angled triangles are similar to each other.

11. The sine, tangent, or secant of an angle, is the sine, tangent, or secant of the arc by which the angle is measured, or of the degrees, &c. in the same arc or angle.

12. The method of constructing the scales of chords, sines, tangents, and secants, usually engraven on instruments, for practice, is exhibited in the annexed figure.



13. A Trigonometrical Canon, is a table showing the length of the sine, tangent, and secant, to every degree and minute of the quadrant, with respect to the radius, which is expressed by unity or 1, with any number of ciphers. The logarithms of these sines, tangents, and secants, are also ranged in the tables;

tables; and these are most commonly used, as they perform the calculations by only addition and subtraction, instead of the multiplication and division by the natural sines, &c, according to the nature of logarithms. Such a table of log. sines and tangents, as well as the logs. of common numbers, are placed at the end of this volume, and the description and use of them are as follow; viz. of the sines and tangents; and the other table, of common logs. has been already explained in the first volume of this Course.

Description of the Table of Log. Sines and Tangents.

In the first column of the table are contained all the arcs, or angles, for every minute in the quadrant, viz. from 1′ to 45°, descending from top to bottom by the left-hand side, and then returning back by the right-hand side, ascending from bottom to top, from 45° to 90°; the degrees being set at top or bottom, and the minutes in the column. Then the sines, cosines, tangents, cotangents, of the degrees and minutes, are placed on the same lines with them, and in the annexed columns, according to their several respective names or titles, which are at the top of the columns for the degrees at the top, but at the bottom of the columns for the degrees at the bottom of the leaves. The secants and cosecants are omitted in this table, because they are so easily found from the sines and cosines; for, of every arc or angle, the sine and cosecant together make up 20 or double the radius, and the cosine and secant together make up the same 20 also. Therefore, if a secant is wanted, we have only to subtract the cosine from 20; or, to find the cosecant, take the sine from 20. And the best way to perform these subtractions, because it may be done at sight, is to begin at the left hand, and take every figure from 9, but the last or right hand figure from 10, prefixing 1, for 10, before the first figure of the remainder.

PROBLEM I.

To compute the Natural Sine and Cosine of a Given Arc.

THIS problem is resolved after various ways. One of these is as follows, viz. by means of the ratio between the diameter and circumference of a circle, together with the known series for the sine and cosine, hereafter demonstrated. Thus, the semi-

semicircumference of the circle, whose radius is 1, bei
3·14159265358979 &c, the proportion will therefore b
as the number of degrees or minutes in the simicircle,
is to the degrees or minutes in the proposed arc,
so is 3·14159265 &c, to the length of the said arc.

This length of the arc being denoted by the letter a; a its sine and cosine by s and c; then will these two be e pressed by the two following series, viz.

$$s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7} + \&c.$$
$$= a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \&c.$$
$$c = 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2.3.4.5.6} + \&c.$$
$$= 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720} + \&c.$$

EXAM. 1. If it be required to find the sine and cosine 1 minute. Then, the number of minutes in 180° be 10800, it will be first, as 10800 : 1 :: 3·14159265 &c ·000290888208665 = the length of an arc of one minu Therefore, in this case,

$a =$ ·0002908882
and $\frac{1}{6}a^3 =$ ·000000000004 &c,
the diff. is $s =$ ·0002908882 the sine of 1 minute.
Also, from 1.
take $\frac{1}{2}a^2 =$ 0·0000000423079 &c,
leaves $c =$ ·9999999577 the cosine of 1 minu

EXAM. 2. For the sine and cosine of 5 degrees. Here, as 180° : 5° :: 3·14159265 &c. : ·08726646 = a length of 5 degrees. Hence $a =$ ·08726646
$-\frac{1}{6}a^3 = -$ ·00011076
$+\frac{1}{120}a^5 =$ ·00000004

these collected give $s =$ ·08715574 the sine of

And, for the cosine, $1 =$ 1·
$-\frac{1}{2}a^2 = -$ ·00380771
$+\frac{1}{24}a^4 =$ ·00000241

these collected give $c =$ ·99619470 the cosine of 5°.

After the same manner, the sine and cosine of any ot arc may be computed. But the greater the arc is, the slo

the series will converge, in which case a greater number of terms must be taken, to bring out the conclusion to the same degree of exactness.

Or, having found the sine, the cosine will be found from it, by the property of the right-angled triangle CBF, viz. the cosine $CF = \sqrt{CB^2 - BF^2}$, or $c = \sqrt{1 - s^2}$.

There are also other methods of constructing the canon of sines and cosines, which, for brevity's sake, are here omitted.

PROBLEM II.

To compute the Tangents and Secants.

THE sines and cosines being known, or found by the foregoing problem; the tangents and secants will be easily found, from the principle of similar triangles, in the following manner:

In the first figure, where, of the arc AB, BF is the sine, CF or BK the cosine, AH the tangent, CH the secant, DL the cotangent, and CL the cosecant, the radius being CA or CB or CD; the three similar triangles CFB, CAH, CDL, give the following proportions:

1*st*, CF : FB :: CA : AH; whence the tangent is known, being a fourth proportional to the cosine, sine, and radius.

2*d*, CF : CB :: CA : CH; whence the secant is known, being a third proportional to the cosine and radius.

3*d*, BF : FC :: CD : DL; whence the cotangent is known, being a fourth proportional to the sine, cosine, and radius.

4*th*, BF : BC :: CD : CL; whence the cosecant is known, being a third proportional to the sine and radius.

As for the log. sines, tangents, and secants, in the tables, they are only the logarithms of the natural sines, tangents, and secants, calculated as above.

HAVING given an idea of the calculation and use of sines, tangents and secants, we may now proceed to resolve the several cases of Trigonometry; previous to which, however, it may be proper to add a few preparatory notes and observations, as below.

Note 1. There are usually three methods of resolving triangles, or the cases of trigonometry; namely, Geometrical Construction, Arithmetical Computation, and Instrumental Operation.

In the First Method, The triangle is constructed, by making the parts of the given magnitudes, namely, the sides from a scale of equal parts, and the angles from a scale of chords,

or

or by some other instrument. Then measuring the unknown parts by the same scales or instruments, the solution will be obtained near the truth.

In the Second Method, Having stated the terms of the proportion according to the proper rule or theorem, resolve it like any other proportion, in which a fourth term is to be found from three given terms, by multiplying the second and third together, and dividing the product by the first, in working with the natural numbers; or, in working with the logarithms, add the logs. of the second and third terms together, and from the sum take the log. of the first term; then the natural number answering to the remainder is the fourth term sought.

In the Third Method, Or Instrumentally, as suppose by the log. lines on one side of the common two-foot scales; Extend the Compasses from the first term, to the second or third, which happens to be of the same kind with it; then that extent will reach from the other term to the fourth term, as required, taking both extents towards the same end of the scale.

Note 2. Every triangle has six parts, viz. three sides and three angles. And in every triangle, or case in trigonometry, there must be given three of these parts, to find the other three. Also, of the three parts that are given, one of them at least must be a side; because, with the same angles, the sides may be greater or less in any proportion.

Note 3. All the cases in trigonometry, may be comprised in three varieties only; viz.

1*st*, When a side and its opposite angle are given.

2*d*, When two sides and the contained angle are given.

3*d*, When the three sides are given.

For there cannot possibly be more than these three varieties of cases; for each of which it will therefore be proper to give a separate theorem, as follows:

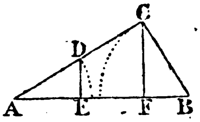
THEOREM I.

When a Side and its Opposite Angle are two of the Given Parts.

THEN the unknown parts will be found by this theorem; viz. The sides of the triangle have the same proportion to each other, as the sines of their opposite angles have.

That is, As any one side,
Is to the sine of its opposite angle;
So is any other side,
To the sine of its opposite angle.

Demonstr.

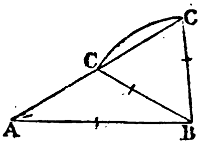


Demonstr. For, let ABC be the proposed triangle, having AB the greatest side, and BC the least. Take AD = BC, considering it as a radius; and let fall the perpendiculars DE, CF, which will evidently be the sines of the angles A and B, to the radius AD or BC. Now the triangles ADE, ACF, are equiangular; they therefore have their like sides proportional, namely, AC : CF : : AD or BC : DE; that is, the side AC is to the sine of its opposite angle B, as the side BC is to the sine of its opposite angle A.

Note 1. In practice, to find an angle, begin the proportion with a side opposite to a given angle. And to find a side, begin with an angle opposite to a given side.

Note 2. An angle found by this rule is ambiguous, or uncertain whether it be acute or obtuse, unless it be a right angle, or unless its magnitude be such as to prevent the ambiguity; because the sine answers to two angles, which are supplements to each other; and accordingly the geometrical construction forms two triangles with the same parts that are given, as in the example below; and when there is no restriction or limitation included in the question, either of them may be taken. The number of degrees in the table, answering to the sine, is the acute angle; but if the angle be obtuse, subtract those degrees from 180°, and the remainder will be the obtuse angle. When a given angle is obtuse, or a right one, there can be no ambiguity; for then neither of the other angles can be obtuse, and the geometrical construction will form only one triangle.

EXAMPLE I.



In the plane triangle ABC,

Given $\begin{cases} \text{AB } 345 \text{ yards} \\ \text{BC } 232 \text{ yards} \\ \angle \text{ A } 37^\circ\ 20' \end{cases}$

Required the other parts.

1. *Geometrically.*

Draw an indefinite line; on which set off AB = 345, from some convenient scale of equal parts.—Make the angle A = $37^\circ\frac{1}{3}$.—With a radius of 232, taken from the same scale of equal parts, and centre B, cross AC in the two points C, C.—Lastly, join BC, BC, and the figure is constructed,

structed, which gives two triangles, and showing that the case is ambiguous.

Then, the sides AC measured by the scale of equal parts, and the angles B and C measured by the line of chords, or other instrument, will be found to be nearly as below; viz.

AC 174	∠ B 27°	∠ C 115°$\frac{1}{2}$.
or 374$\frac{1}{2}$	or 78$\frac{1}{4}$	or 64$\frac{1}{2}$.

2. *Arithmetically.*

First, to find the angles at C.

As side	BC	232	- - log.	2·365488
To sin. op.	∠ A	37° 20′	- -	9·782796
So side	AB	345	- -	2·537819
To sin. op.	∠ C	115° 36′ or 64° 24′		9·955127
add	∠ A	37 20	37 20	
the sum		152 56 or 101 44		
taken from		180 00	180 00	
leaves	∠ B	27 04 or 78 16		

Then, to find the side AC.

As sine ∠ A	37° 20′	- -	log. 9·782796
To op. side BC	232	- -	2·365488
So sin. ∠ B	27° 04′	- -	9·658037
	78 16	- -	9·990829
To op. side AC	174·07	- -	2·240729
or	374·56	- -	2·573521

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 232 to 345 on the line of numbers; then that extent reaches, on the sines, from 37°$\frac{1}{3}$ to 64°$\frac{1}{2}$, the angle C.

In the second proportion.—Extend the compasses from 37°$\frac{1}{3}$ to 27° or 78°$\frac{1}{4}$, on the sines; then that extent reaches, on the line of numbers, from 232 to 174 or 374$\frac{1}{2}$, the two values of the side AC.

EXAMPLE II.

In the plane triangle ABC,

$$\text{Given}\begin{cases}\text{AB } 365 \text{ poles}\\ \angle \text{A } 57^\circ\ 12'\\ \angle \text{B } 24\ \ 45\end{cases}\qquad \text{Ans.}\begin{cases}\angle \text{C } 98^\circ\ 3'\\ \text{AC } 154{\cdot}33\\ \text{BC } 309{\cdot}86\end{cases}$$

Required the other parts.

EXAMPLE

EXAMPLE III.

In the plane triangle ABC,

Given $\begin{cases} \text{AC} & 120 \text{ feet} \\ \text{BC} & 112 \text{ feet} \\ \angle \text{A} & 57^\circ\ 28' \end{cases}$ Ans. $\begin{cases} \angle \text{B} & 64^\circ\ 35' \\ \text{or} & 115\ \ 25 \\ \angle \text{C} & 57\ \ 57 \\ \text{or} & 7\ \ \ 7 \\ \text{AB} & 112{\cdot}6 \text{ feet} \\ \text{or} & 16{\cdot}47 \text{ feet.} \end{cases}$

Required the other parts.

THEOREM II.

When two Sides and their Contained Angle are given.

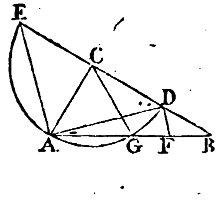
FIRST add the two given sides together, to get their sum, and subtract them, to get their difference. Next subtract the given angle from 180°, or two right angles, and the remainder will be the sum of the two other angles; then divide that by 2, which will give the half sum of the said unknown angles. Then say,

As the sum of the two given sides,
Is to the difference of the same sides;
So is the tang. of half the sum of their op. angles,
To the tang. of half the diff. of the same angles.

Then add the half difference of the angles, so found, to their half-sum, and it will give the greater angle, and subtracting the same will leave the less angle: because the half sum of any two quantities, increased by their half difference, gives the greater, and diminished by it gives the less.

Then all the angles being now known, the unknown side will be found by the former theorem.

Note. Instead of the tangent of the half sum of the unknown angles, in the third term of the proportion, may be used the cotangent of half the given angle, which is the same thing.



Demonst. Let ABC be the proposed triangle, having the two given sides AC, BC, including the given angle C. With the centre C, and radius CA, the less of these two sides, describe a semicircle, meeting the other side BC produced in D, E, and the unknown side AB in A, G. Join AE, AD, CG, and draw DF parallel to AE.

Then BE *is the sum of* the two given sides AC, CB, or of EC, CB; *and* BD *is the difference* of the same two given sides

AC,

AC, BC, or of CD, CB. Also, the external angle ACE, is equal to the given sum of the two internal angles CAB, CBA; but the angle ADE, at the circumference, is equal to half the angle ACE at the centre; therefore the same angle ADE is equal to half the given sum of the angles CAB, CBA. Also, the external angle AGC, of the triangle BCG, is equal to the sum of the two internal angles GCB, GBC, or the angle GCB is equal to the difference of the two angles AGC, GBC; but the angle CAB is equal to the said angle AGC, these being opposite to the equal sides AC, CG; and the angle DAB, at the circumference, is equal to half the angle DCG at the centre; therefore the angle DAB is equal to half the difference of the two angles CAB, CBA; of which it has been shown that ADE or CDA is the half sum.

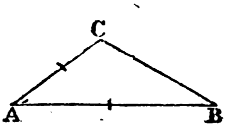
Now the angle DAE, in a semicircle, is a right angle, or AE is perpendicular to AD; and DF, parallel to AE, is also perpendicular to AD: consequently AE is the tangent of CDA the half sum, and DF the tangent of DAB the half difference of the angles, to the same radius AD, by the definition of a tan gent. But the tangents AE, DF being parallel, it will be, as BE : BD : : AE : DF; that is, as the sum of the sides is to the difference of the sides, so is the tangent of half the sum of the opposite angles, to the tangent of half their difference.

EXAMPLE I.

In the plane triangle ABC,

Given $\begin{cases} \text{AB } 345 \text{ yards} \\ \text{AC } 174{\cdot}07 \text{ yards} \\ \angle \text{A } 37^\circ\ 20' \end{cases}$

Required the other parts.



1. *Geometrically.*

Draw AB = 345 from a scale of equal parts. Make the angle A = 37° 20′. Set off AC = 174 by the scale of equal parts. Join BC, and it is done.

Then the other parts being measured, they are found to be nearly as follow; viz. the side BC 232 yards, the angle B 27°, and the angle C 115°½.

2. *Arithmetically.*

The side AB	345	From	180°	00′
the side AC	174·07	take ∠ A	37	20
their *sum*	*519·07*	sum of C and B	142	40
their differ.	*170·93*	half sum of do.	71	20

As sum of sides AB, AC,	519·07	log.	2·715226
To diff. of sides AB, AC,	170·93	-	2·232818
So tang. half sum ∠s C and B	71° 20	-	10·471298
To tang. half diff. ∠s C and B	44 16	-	9·988890
these added give ∠ C	115 36		
and subtr. give ∠ B	27 4		

Then, by the former theorem,

As sin. ∠ C 115° 36′ or 64° 24′	log.	9·955126
To its op. side AB 345	-	2·537819
So sin. of ∠ A 37° 20′	-	9·782796
To its op. side BC 232	-	2·365489

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 519 to 171, on the line of numbers; then that extent reaches, on the tangents, from 71°$\frac{1}{3}$ (the contrary way, because the tangents are set back again from 45°) a little beyond 45, which being set so far back from 45, falls upon 44°$\frac{1}{4}$, the fourth term.

In the second proportion.—Extend from 64°$\frac{1}{3}$ to 37°$\frac{1}{3}$, on the sines; then that extent reaches, on the numbers, from 345 to 232, the fourth term sought.

EXAMPLE II.

In the plane triangle ABC,

Given { AB 365 poles, AC 154·33, ∠ A 57° 12′ } Ans. { BC 309·86, ∠ B 24° 45′, ∠ C 98 3 }

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given { AC 120 yards, BC 112 yards, ∠ C 57° 57′ } Ans. { AB 112·6, ∠ A 57° 28, ∠ B 64 35 }

Required the other parts.

THEOREM III.

When the Three Sides of a Triangle are given.

First, let fall a perpendicular from the greatest angle on the opposite side, or base, dividing it into two segments, and the whole triangle into two right-angled triangles: then the proportion will be,

As

As the base, or sum of the segments,
Is to the sum of the other two sides;
So is the difference of those sides,
To the diff. of the segments of the base.

Then take half this difference of the segments, and add it to the half sum, or the half base, for the greater segment; and subtract the same for the less segment.

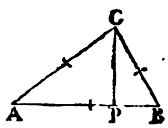
Hence, in each of the two right-angled triangles, there will be known two sides, and the right angle opposite to one of them; consequently the other angles will be found by the first theorem.

Demonstr. By theor. 35, Geom. the rectangle of the sum and difference of the two sides, is equal to the rectangle of the sum and difference of the two segments. Therefore, by forming the sides of these rectangles into a proportion by theor. 76, Geometry, it will appear that the sums and differences are proportional as in this theorem.

EXAMPLE I.

In the plane triangle ABC,

Given the sides $\begin{cases} \text{AB } 345 \text{ yards} \\ \text{AC } 232 \\ \text{BC } 174\cdot07 \end{cases}$



To find the angles.

1. *Geometrically.*

Draw the base AB = 345 by a scale of equal parts. With radius 232, and centre A, describe an arc; and with radius 174, and centre B, describe another arc, cutting the former in C. Join AC, BC, and it is done.

Then, by measuring the angles, they will be found to be nearly as follows, viz.

$\angle$ A $27°$, $\angle$ B $37°\frac{1}{2}$, and $\angle$ C $115°\frac{1}{2}$.

2. *Arithmetically.*

Having let fall the perpendicular CP, it will be,
As the base AB : AC + BC : : AC − BC : AP − BP,
that is, as 345 : 406·07 : : 57·93 : 68·18 = AP − BP,

its half is -	34·09
the half base is	172·50
the sum of these is	206·59 = AP
and their diff. is	138·41 = BP

Then,

Then, in the triangle APC, right-angled at P,

As the side	AC	- - 232	-	log. 2·365488
To sin. op.	∠ P	- - 90°	-	10·000000
So is the side	AP	- - 206·59	- -	2·315109
To sin. op. ∠	ACP	- - 62° 56′	-	9·949621
Which taken from		- 90 00		
leaves the ∠ A		27 04		

Again, in the triangle BPC, right angled at P,

As the side	BC	- - 174·07	-	log. 2·240724
To sin. op.	∠ P	- - 90°	-	10·000000
So is side	BP	- - 138·41	- -	2·141168
To sin. op. ∠	BCP	- - 52° 40′	- -	9·900444
which taken from		- 90 00		
leaves the ∠ B		37 20		

Also, the	∠ ACP	62° 56′
added to	∠ BCP	52 40
gives the whole	∠ ACB	115 36

So that all the three angles are as follow, viz.

the ∠ A 27° 4′; the ∠ B 37° 20′; the ∠ C 115° 36′.

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 345 to 406, on the line of numbers; then that extent reaches, on the same line, from 58 to 68·2 nearly, which is the difference of the segments of the base.

In the second proportion.—Extend from 232 to $206\frac{1}{2}$, on the line of numbers; then that extent reaches, on the sines, from 90° to 63°.

In the third proportion.—Extend from 174 to $138\frac{1}{2}$; then that extent reaches from 90° to $52°\frac{2}{3}$ on the sines.

EXAMPLE II.

In the plane triangle ABC,

Given the sides $\begin{cases} \text{AB } 365 \text{ poles} \\ \text{AC } 154·33 \\ \text{BC } 309·86 \end{cases}$ Ans. $\begin{cases} \angle \text{ A } 57° \; 12′ \\ \angle \text{ B } 24 \; 45 \\ \angle \text{ C } 93 \; 3 \end{cases}$

To find the angles.

EXAMPLE

EXAMPLE III.

In the plane triangle ABC,

Given the sides $\begin{cases} AB\ 120 \\ AC\ 112\cdot6 \\ BC\ 112 \end{cases}$ Ans. $\begin{cases} \angle A\ 57^\circ\ 28' \\ \angle B\ 57\ \ 57 \\ \angle C\ 64\ \ 35 \end{cases}$

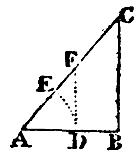
To find the angles.

The three foregoing theorems include all the cases of plane triangles, both right-angled and oblique. But there are other theorems suited to some particular forms of triangles, which are sometimes more expeditious in their use than the general ones; one of which, as the case for which it serves so frequently occurs, may be here taken, as follows:

THEOREM IV.

When a Triangle is Right-angled; any of the unknown parts may be found by the following proportions: viz.

As radius
Is to either leg of the triangle;
So is tang. of its adjacent angle,
To its opposite leg;
And so is secant of the same angle,
To the hypothenuse.



Demonstr. AB being the given leg, in the right-angled triangle ABC; with the centre A, and any assumed radius AD, describe an arc DE, and draw DF perpendicular to AB, or parallel to BC. Then it is evident, from the definitions, that DF is the tangent, and AF the secant of the arc DE, or of the angle A which is measured by that arc, to the radius AD. Then, because of the parallels BC, DF, it will be, - - - as AD : AB : : DF : BC and : : AF : AC, which is the same as the theorem is in words.

Note. The radius is equal, either to the sine of 90°, or the tangent of 45°; and is expressed by 1, in a table of natural sines, or by 10 in the log. sines.

EXAMPLE I.

In the right-angled triangle ABC,

Given $\begin{Bmatrix} \text{the leg AB } 162 \\ \angle A\ 53^\circ\ 7'\ 48'' \end{Bmatrix}$ To find AC and BC.

1. *Geometrically.*

1. *Geometrically.*

Make AB = 162 equal parts, and the angle A = 53° 7′ 48″; then raise the perpendicular BC, meeting AC in C. So shall AC measure 270, and BC 216.

2. *Arithmetically.*

As radius	-		-	log. 10·000000
To leg AB	-	162	-	2·209515
So tang. ∠ A	-	53° 7′ 48″	-	10·124937
To leg. BC	-	216	-	2·334452
So secant ∠ A	-	53° 7′ 48″	-	10·221848
To hyp. AC	-	270	-	2·431363

3. *Instrumentally.*

Extend the compasses from 45° to 53°$\frac{1}{8}$, on the tangents. Then that extent will reach from 162 to 216 on the line of numbers.

EXAMPLE II.

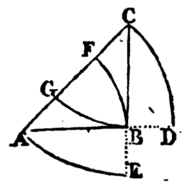
In the right-angled triangle ABC,

Given $\begin{cases}\text{the leg AB } 180 \\ \text{the } \angle \text{ A } 62^\circ\ 40'\end{cases}$ Ans. $\begin{cases}\text{AC } 392{\cdot}0146 \\ \text{BC } 348{\cdot}2464\end{cases}$

To find the other two sides.

Note. There is sometimes given another method for right-angled triangles, which is this:

ABC being such a triangle, make one leg AB radius; that is, with centre A, and distance AB, describe an arc BF. Then it is evident that the other leg BC represents the tangent, and the hypothenuse AC the secant, of the arc BF, or of the angle A.



In like manner, if the leg BC be made radius; then the other leg AB will represent the tangent, and the hypothenuse AC the secant, of the arc BG or angle C.

But if the hypothenuse be made radius; then each leg will represent the sine of its opposite angle; namely, the leg AB the sine of the arc AE or angle C, and the leg BC the sine of the arc CD or angle A.

Then the general rule for all these cases is this, namely, that the sides of the triangle bear to each other the same proportion as the parts which they represent.

And this is called, Making every side radius.

Note

Note 2. When there are given two sides of a right-angled triangle, to find the third side; this is to be found by the property of the squares of the sides, in theorem 34, Geom. viz. that the square of the hypothenuse, or longest side, is equal to both the squares of the two other sides together. Therefore, to find the longest side, add the squares of the two shorter sides together, and extract the square root of that sum; but to find one of the shorter sides, subtract the one square from the other, and extract the root of the remainder.

OF HEIGHTS AND DISTANCES, &c.

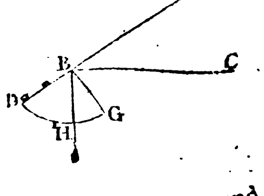
BY the mensuration and protraction of lines and angles, are determined the lengths, heights, depths, and distances of bodies or objects.

Accessible lines are measured by applying to them some certain measure a number of times, as an inch, or a foot, or yard. But inaccessible lines must be measured by taking angles, or by such-like method, drawn from the principles of geometry.

When instruments are used for taking the magnitude of the angles in degrees, the lines are then calculated by trigonometry: in the other methods, the lines are calculated from the principle of similar triangles, or some other geometrical property, without regard to the measure of the angles.

Angles of elevation, or of depression, are usually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furnished with a plummet suspended from the centre, and two open sights fixed on one of the radii, or else with telescopic sights.

To take an Angle of Altitude and Depression with the Quadrant.

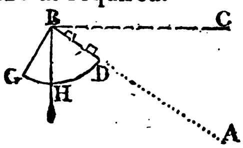


Let A be any object, as the sun, moon, or a star, or the top of a tower, or hill, or other eminence: and let it be required to find the measure of the angle ABC, which a line drawn from the object makes above the horizontal line BC.

Place the centre of the quadrant in the angular point, and move it round

round there as a centre, till with one eye at D, the other being shut, you perceive the object A through the sights; then will the arc GH of the quadrant, cut off by the plumb-line BH, be the measure of the angle ABC as required.

The angle ABC of depression of any object A, below the horizontal line BC, is taken in the same manner; except that here the eye is applied to the centre, and the measure of the angle is the arc GH, on the other side of the plumb-line.



The following examples are to be constructed and calculated by the foregoing methods, treated of in Trigonometry.

EXAMPLE I.

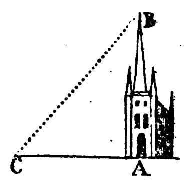
Having measured a distance of 200 feet, in a direct horizontal line, from the bottom of a steeple, the angle of elevation of its top, taken at that distance, was found to be 47° 30′; hence it is required to find the height of the steeple.

Construction.

Draw an indefinite line; on which set off AC = 200 equal parts, for the measured distance. Erect the indefinite perpendicular AB; and draw CB so as to make the angle C = 47° 30′, the angle of elevation; and it is done. Then AB, measured on the scale of equal parts, is nearly $218\frac{1}{4}$.

Calculation.

As radius	- -	10·000000
To AC 200	- -	2·301030
So tang. ∠ C 47° 30′		10·037948
To AB 218·26 required		2·338978



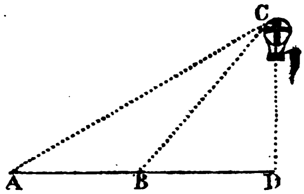
EXAMPLE II.

What was the perpendicular height of a cloud, or of a balloon, when its angles of elevation were 35° and 64°, as taken by two observers, at the same time, both on the same side of it, and in the same vertical plane; the distance between them being half a mile or 880 yards. And what was its distance from the said two observers?

Construction.

Construction.

Draw an indefinite ground line, on which set off the given distance AB = 880; then A and B are the places of the observers. Make the angle A = 35°, and the angle B = 64°; then the intersection of the lines at C will be the place of the balloon: whence the perpendicular CD, being let fall, will be its perpendicular height. Then by measurement are found the distances and height nearly as follow, viz. AC 1631, BC 1041, DC 936.



Calculation.

First, from ∠ B		64°
	take ∠ A	35
	leaves ∠ ACB	29

Then in the triangle ABC,

As sin. ∠ ACB	29°	-	-	-	9·685571
To op. side AB	880	-	-	-	2·944483
So sin. ∠ A	35°	-	-	-	9·758591
To op. side BC	1041·125	-	-	-	3·017503
As sin. ∠ ACB	29°	-	-	-	9·685571
To op. side AB	880	-	-	-	2·944483
So sin. ∠ B 116° or 64°		-	-	-	9·953660
To op. side AC	1631·442	-	-	-	3·212572

And in the triangle BCD,

As sin. ∠ D	90°	-	-	-	10·000000
To op. side BC	1041·125	-	-	-	3·017503
So sin. ∠ B	64°	-	-	-	9·953660
To op. side CD	935·757	-	-	-	2·971163

EXAMPLE III.

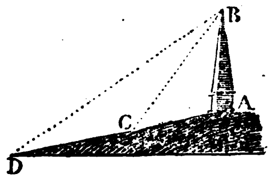
Having to find the height of an obelisk standing on the top of a declivity, I first measured from its bottom a distance of 40 feet, and there found the angle, formed by the oblique plane and a line imagined to go to the top of the obelisk, 41°; but after measuring on in the same direction 60 feet farther, the like angle was only 23° 45′. What then was the height of the obelisk?

 Construction.

Construction.

Draw an indefinite line for the sloping plane or declivity, in which assume any point A for the bottom of the obelisk, from which set off the distance AC = 40, and again CD = 60 equal parts. Then make the angle C = 41°, and the angle D = 23° 45′; and the point B where the two lines meet will be the top of the obelisk. Therefore AB, joined, will be its height.

Calculation.



From the ∠ C	41°	00′
take the ∠ D	23	45
leaves the ∠ DBC	17	15

Then in the triangle DBC,

As sin. ∠ DBC 17° 15′	- - -	9·472086
To op. side DC 60	- - - -	1·778151
So sin. ∠ D 23 45	- - -	9·605032
To op. side CB 81·488	- - -	1·911097

And in the triangle ABC,

As sum of sides CB, CA	121·488	-	2·084533
To diff. of sides CB, CA	41·488	-	1·617923
So tang. half sum ∠s A, B	69° 30′	-	10·427262
To tang. half diff. ∠s A, B	42 24½	-	9·960652
the diff. of these is ∠ CBA	27 5½		

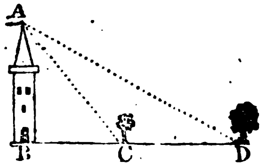
Lastly, as sin. ∠ CBA	27° 5′½	- -	9·658284
To op. side CA	40	- -	1·602060
So sin. ∠ C -	41° 0′	- -	9·816943
To op. side AB	57·623	- -	1·760719

EXAMPLE IV.

Wanting to know the distance between two inaccessible trees, or other objects, from the top of a tower 120 feet high, which lay in the same right line with the two objects, I took the angles formed by the perpendicular wall and lines conceived to be drawn from the top of the tower to the bottom of each tree, and found them to be 33° and 64°½. What then may be the distance between the two objects?

Construction.

Construction.



Draw the indefinite ground line BD, and perpendicular to it BA = 120 equal parts. Then draw the two lines AC, AD, making the two angles BAC, BAD, equal to the given measures 33° and 64°½. So shall C and D be the places of the two objects.

Calculation.

First, in the right-angled triangle ABC,

As radius	- - - - - -	10·000000
To AB	- 120 - - -	2·079181
So tang. ∠ BAC	33° - - -	9·812517
To BC	- 77·929 - - -	1·891698

Then in the right-angled triangle ABD,

As radius	- - - - -	10·000000
To AB	- - 120 - -	2·079181
So tang. ∠ BAD	- 64°½ - -	10·321504
To BD	- 251·585 - -	2·400685

From which take BC 77·929
leaves the dist. CD 173·656 as required.

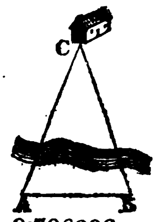
EXAMPLE V.

Being on the side of a river, and wanting to know the distance to a house which was seen on the other side, I measured 200 yards in a straight line by the side of the river; and then, at each end of this line of distance, took the horizontal angle formed between the house and the other end of the line; which angles were, the one of them 68° 2′, and the other 73° 15′. What then were the distances from each end to the house?

Construction.

Draw the line AB = 200 equal parts. Then draw AC so as to make the angle A = 68° 2′, and BC to make the angle B = 73° 15′. So shall the point C be the place of the house required.

Calculation.



Calculation:

	To the given ∠ A	68° 2′		
	add the given ∠ B	73 15		
	then their sum	141 17		
	being taken from	180 0		
	leaves the third ∠ C	38 43		
Hence,	As sin. ∠ C	38° 43′	-	9·796206
	To op. side AB	200	-	2·301030
	So sin. ∠ A	68° 2′	-	9·967268
	To op. side BC	296·54	-	2·472092
And,	As sin. ∠ C	38° 43′	-	9·796206
	To op. side AB	200	-	2·301030
	So sin. ∠ B	73° 15′	-	9·981171
	To op. side AC	306·19	-	2·485995

EXAM. VI. From the edge of a ditch, of 36 feet wide, surrounding a fort, having taken the angle of elevation of the top of the wall, it was found to be 62° 40′: required the height of the wall, and the length of a ladder to reach from my station to the top of it? Ans. {height of wall 69·64, ladder, 78·4 feet.

EXAM. VII. Required the length of a shoar, which being to strut 11 feet from the upright of a building, will support a jamb 23 feet 10 inches from the ground?

Ans. 26 feet 3 inches.

EXAM. VIII. A ladder, 40 feet long, can be so planted, that it shall reach a window 33 feet from the ground, on one side of the street; and by turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high, on the other side: required the breadth of the street? Ans. 56·649 feet.

EXAM. IX. A maypole, whose top was broken off by a blast of wind, struck the ground at 15 feet distance from the foot of the pole: what was the height of the whole maypole, supposing the broken piece to measure 39 feet in length?

Ans. 75 feet.

EXAM. X. At 170 feet distance from the bottom of a tower, the angle of its elevation was found to be 52° 30′: required the altitude of the tower? Ans. 221·55 feet.

EXAM. XI. From the top of a tower, by the sea-side, of 143 feet high, it was observed that the angle of depression of a ship's bottom, then at anchor, measured 35°; what then was the ship's distance from the bottom of the wall?

Ans. 204·22 feet.

EXAM.

EXAM. XII. What is the perpendicular height of a hill; its angle of elevation, taken at the bottom of it, being 46°, and 200 yards farther off, on a level with the bottom, the angle was 31°? Ans. 286·28 yards.

EXAM. XIII. Wanting to know the height of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°: required its height, and my distance from it at the first station?

Ans. $\begin{cases}\text{height} & 307{\cdot}53\\ \text{distance} & 192{\cdot}15\end{cases}$

EXAM. XIV. Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill; I took the angle of elevation of the top of the hill 40°, and of the top of the tower 51°; then measuring in a line directly from it to the distance of 200 feet farther, I found the angle to the top of the tower to be 33° 45′. What then is the height of the tower?

Ans. 93·33148 feet.

EXAM. XV. From a window near the bottom of a house, which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′: what then is the height and distance of the steeple?

Ans. $\begin{cases}\text{height} & 210{\cdot}44\\ \text{distance} & 250{\cdot}79\end{cases}$

EXAM. XVI. Wanting to know the height of, and my distance from, an object on the other side of a river, which appeared to be on a level with the place where I stood, close by the side of the river; and not having room to measure backward, in the same line, because of the immediate rise of the bank, I placed a mark where I stood, and measured in a direction from the object, up the ascending ground, to the distance of 264 feet, where it was evident that I was above the level of the top of the object; there the angles of depression were found to be, viz. of the mark left at the river's side 42°, of the bottom of the object 27°, and of its top 19°. Required then the height of the object, and the distance of the mark from its bottom?

Ans. $\begin{cases}\text{height} & 57{\cdot}26\\ \text{distance} & 150{\cdot}50\end{cases}$

EXAM. XVII. If the height of the mountain called the Peak of Teneriffe be $2\frac{1}{2}$ miles, as it is nearly, and the angle taken

taken at the top of it, as formed between a plumb-line and a line conceived to touch the earth in the horizon, or farthest visible point, be 87° 58′; it is required from these measures to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly globular?

$$\text{Ans.} \begin{cases} \text{dist.} & 140 \cdot 876 \\ \text{diam.} & 7936 \end{cases} \text{miles.}$$

EXAM. XVIII. Two ships of war, intending to cannonade a fort, are, by the shallowness of the water, kept so far from it, that they suspect their guns cannot reach it with effect. In order therefore to measure the distance, they separate from each other a quarter of a mile, or 440 yards; then each ship observes and measures the angle which the other ship and the fort subtends, which angles are 83° 45′ and 85° 15′. What then is the distance between each ship and the fort?

$$\text{Ans.} \begin{cases} 2292 \cdot 26 \text{ yards,} \\ 2298 \cdot 05 \end{cases}$$

EXAM. XIX. Being on the side of a river, and wanting to know the distance to a house which was seen at a distance on the other side; I measured out for a base 400 yards in a right line by the side of the river, and found that the two angles, one at each end of this line, subtended by the other end and the house, were 68° 2′ and 73° 15′. What then was the distance between each station and the house?

$$\text{Ans.} \begin{cases} 593 \cdot 08 \text{ yards.} \\ 612 \cdot 38 \end{cases}$$

EXAM. XX. Wanting to know the breadth of a river, I measured a base of 500 yards in a straight line close by one side of it; and at each end of this line I found the angles subtended by the other end and a tree, close to the bank on the other side of the river, to be 53° and 79° 12′. What then was the perpendicular breadth of the river?

Ans. 529·48 yards.

EXAM. XXI. Wanting to know the extent of a piece of water, or distance between two headlands; I measured from each of them to a certain point inland, and found the two distances to be 735 yards and 840 yards; also the horizontal angle subtended between these two lines was 55° 40′. What then was the distance required? Ans. 741·2 yards.

EXAM. XXII. A point of land was observed, by a ship at sea, to bear east-by-south; and after sailing north-east 12 miles, it was found to bear south-east-by-east. It is required

to

to determine the place of that headland, and the ship's distance from it at the last observation? Ans. 26·0728 miles.

EXAM. XXIII. Wanting to know the distance between a house and a mill, which were seen at a distance on the other side of a river, I measured a base line along the side where I was, of 600 yards, and at each end of it took the angles subtended by the other end and the house and mill, which were as follow, viz. at one end the angles were 58° 20′ and 95° 20′, and at the other end the like angles were 53° 30′ and 98° 45′. What then was the distance between the house and mill? Ans. 959·5866 yards.

EXAM. XXIV. Wanting to know my distance from an inaccessible object O, on the other side of a river; and having no instrument for taking angles, but only a chain or cord for measuring distances; from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object O 100 yards, viz. AC and BD each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC 560. What then was the distance of the object O from each station A and B?

Ans. { AO 536·25, BO 500·09

EXAM. XXV. In a garrison besieged are three remarkable objects, A, B, C, the distances of which from each other are discovered by means of a map of the place, and are as follow, viz. AB $266\frac{1}{4}$, AC 530, BC $327\frac{1}{2}$ yards. Now, having to erect a battery against it, at a certain spot without the place, and being desirous to know whether my distances from the three objects be such, as that they may from thence be battered with effect, I took, with an instrument, the horizontal angles subtended by these objects from my station S, and found them to be as follow, viz. the angle ASB 13° 30′, and the angle BSC 29° 50′; required the three distances, SA, SB, SC; the object B being situated nearest to me, and between the two others A and C?

Ans. { SA 757·14, SB 537·10, SC 655·30

EXAM. XXVI. Required the same as in the last example, when the object B is the farthest from my station, but still seen between the two others as to angular position, and those angles being thus, the angle ASB 33° 45′, and BSC 22° 30′, also the three distances, AB 600, AC 800, BC 400 yards?

Ans. { SA $709\frac{1}{2}$, SB $1042\frac{3}{4}$, SC 934

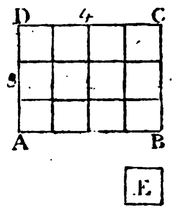
MENSURATION

MENSURATION OF PLANES.

THE Area of any plane figure, is the measure of the space contained within its extremes or bounds; without any regard to thickness.

This area, or the content of the plane figure, is estimated by the number of little squares that may be contained in it; the side of those little measuring squares being an inch, or a foot, or a yard, or any other fixed quantity. And hence the area or content is said to be so many square inches, or square feet, or square yards, &c.

Thus, if the figure to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit proposed: then as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain, which in the present case is 12.



PROBLEM I.

To find the Area of any Parallelogram; whether it be a Square, a Rectangle, a Rhombus, or a Rhomboid.

MULTIPLY the length by the perpendicular breadth, or height, and the product will be the area *.

EXAMPLES.

---

* The truth of this rule is proved in the Geom. theor. 81, cor. 2.

The same is otherwise proved thus: Let the foregoing rectangle be the figure proposed; and let the length and breadth be divided into several parts, each equal to the linear measuring unit, being here 4 for the length, and 3 for the breadth; and let the opposite points of division be connected by right lines.—Then it is evident that these lines divide the rectangle into a number of little squares, each equal to the square measuring unit E; and further, that the number of these little squares, or the area of the figure, is equal to the number of linear measuring units in the length, repeated as often as there are linear measuring

EXAMPLES.

Ex. 1. To find the area of a parallelogram, the length being 12·25, and breadth or height 8·5.

```
12·25 length
  8·5 breadth
-------
 6125
9800
-------
104·125 area.
```

Ex. 2. To find the area of a square, whose side is 35·25 chains. Ans. 124 acres, 1 rood, 1 perch.

Ex. 3. To find the area of a rectangular board, whose length is $12\frac{1}{2}$ feet, and breadth 9 inches. Ans. $9\frac{3}{8}$ feet.

Ex. 4. To find the content of a piece of land, in form of a rhombus, its length being 6·20 chains, and perpendicular breadth 5·45. Ans. 3 acres, 1 rood, 20 perches.

Ex. 5. To find the number of square yards of painting in a rhomboid, whose length is 37 feet, and height 5 feet 3 inches. Ans. $21\frac{7}{12}$ square yards.

PROBLEM II.

To find the Area of a Triangle.

RULE 1. MULTIPLY the base by the perpendicular height, and take half the product for the area *. Or, multiply the one of these dimensions by half the other.

---

measuring units in the breadth, or height; that is, equal to the length drawn into the height; which here is 4 × 3 or 12.

And it is proved (Geom. theor. 25, cor. 2), that any oblique parallelogram is equal to a rectangle, of equal length and perpendicular breadth. Therefore the rule is general for all parallelograms whatever.

* The truth of this rule is evident, because any triangle is the half of a parallelogram of equal base and altitude, by Geom. theor. 26.

EXAMPLES

EXAMPLES.

Ex. 1. To find the area of a triangle, whose base is 625, and perpendicular height 520 links?

Here $625 \times 260 = 162500$ square links,
or equal 1 acre, 2 roods, 20 perches, the answer.

Ex. 2. How many square yards contains the triangle, whose base is 40, and perpendicular 30 feet?

Ans. $66\frac{2}{3}$ square yards.

Ex. 3. To find the number of square yards in a triangle, whose base is 49 feet, and height $25\frac{1}{4}$ feet?

Ans. $68\frac{53}{72}$, or 68·7361.

Ex. 4. To find the area of a triangle, whose base is 18 feet 4 inches, and height 11 feet 10 inches?

Ans. 108 feet, $5\frac{2}{3}$ inches.

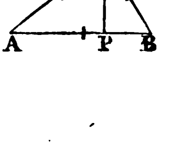
RULE II. When two sides and their contained angle are given: Multiply the two given sides together, and take half their product: Then say, as radius is to the sine of the given angle, so is that half product, to the area of the triangle.

Or, multiply that half product by the natural sine of the said angle, for the area *.

Ex. 1. What is the area of a triangle, whose two sides are 30 and 40, and their contained angle 28° 57′?

By Natural Numbers.	*By Logarithms.*
First, $\frac{1}{2} \times 40 \times 30 = 600$,	
then, $1 : 600 :: \cdot 484046$ sin. 28° 57′	log. 9·684887
600	2·778151
Answer 290·4276 the area answ. to	2·463038

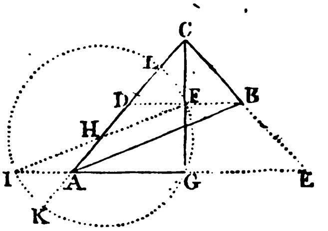
* For, let AB, AC, be the two given sides, including the given angle A. Now $\frac{1}{2}$ AB × CP is the area, by the first rule, CP being the perpendicular. But by trigonometry, as sin. ∠ P, or radius : AC : : sin. ∠ A : CP, which is therefore = AC × sin. ∠ A, taking radius = 1. Therefore the area $\frac{1}{2}$ AB × CP is = $\frac{1}{2}$ AB × AC × sin. ∠ A, to radius 1; or, as radius : sin. ∠ A : : $\frac{1}{2}$ AB × AC : the area.



Ex. 2.

Ex. 2. How many square yards contains the triangle, of which one angle is 45°, and its containing sides 25 and $21\frac{1}{4}$ feet? Ans. 20·86947.

RULE III. When the three sides are given: Add all the three sides together, and take half that sum. Next, subtract each side severally from the said half sum, obtaining three remainders. Then multiply the said half sum and those three remainders all together, and extract the square root of the last product, for the area of the triangle *.



* For, let ABC be the given triangle. Draw the parallels AE, BD, meeting the two sides AC, CB, produced, in D and E, and making CD = CB, and CE = CA. Also draw CFG bisecting DB and AE perpendicularly in F and G; and FHI parallel to the side AB, meeting AC in H, and AE produced in I. Lastly, with centre H, and radius HF, describe a circle meeting AC produced in K; which will pass through G, because G is a right angle, and through I, because, by means of the parallels, AI = FB = DF, therefore HD = HA, and HF = HI = $\frac{1}{2}$AB.

Hence HA or HD is half the difference of the sides AC, CB, and HC = half their sum or = $\frac{1}{2}$AC + $\frac{1}{2}$CB; also HK = HI = $\frac{1}{2}$IF or $\frac{1}{2}$AB; conseq. CK = $\frac{1}{2}$AC + $\frac{1}{2}$CB + $\frac{1}{2}$AB half the sum of all the three sides of the triangle ABC, or CK = $\frac{1}{2}$s, calling s the sum of those three sides. Again HK = HI = $\frac{1}{2}$IF = $\frac{1}{2}$AB, or KL = AB; theref. CL = CK − KL = $\frac{1}{2}$s − AB, and AK = CK − CA = $\frac{1}{2}$s − AC, and AL = DK = CK − CD = $\frac{1}{2}$s − CB.

Now, by the first rule, AG . CG = the △ ACE, and AG . FG = the △ ABE, theref. AG . CF = △ ABC. Also by the parallels, AG : CG : : DF or IA : CF, theref. AG . CF = (△ACB =) CG . IA = CG . DF, conseq. AG . CF . CG . DF = $\triangle^2$ACB.

But CG.CF = CK . CL = $\frac{1}{2}$s . ($\frac{1}{2}$s − AB), and AG . DF = AK . AL = ($\frac{1}{2}$s − AC) . ($\frac{1}{2}$s − BC); theref. AG . CF . CG . DF = $\triangle^2$ ACB = $\frac{1}{2}$s . ($\frac{1}{2}$s—AB).($\frac{1}{2}$s—AC). $\frac{1}{2}$s—BC) is the square of the area of the triangle ABC. Q. E. D.

Otherwise.

Because the rectangle AG . CF = the △ ABC, and since CG : AG : : CF : DF, drawing the first and second terms into CF, and the third and fourth into AG, the propor. becomes CG . CF : AG . CF : : AG . CF : AG . DF, or CG. CF : △ ABC : : △ ABC : BG . DF, that is, the △ABC is a mean proportional between CG . CF and AG . DF, or between $\frac{1}{2}$s.($\frac{1}{2}$s — AB) and ($\frac{1}{2}$s—AC) . ($\frac{1}{2}$s—BC).

Q. E. D.

Ex. 1.

Ex. 1. To find the area of the triangle whose three sides are 20, 30, 40.

20	45	45	45
30	20	30	40
40	—	—	—
——	25 1st rem.	15 2d rem.	5 3d rem.
2) 90	—	—	—
45 half sum			

Then $45 \times 25 \times 15 \times 5 = 84375$,
The root of which is 290·4737, the area.

Ex. 2. How many square yards of plastering are in a triangle, whose sides are 30, 40, 50 feet? Ans. $66\frac{2}{3}$.

Ex. 3. How many acres, &c. contains the triangle, whose sides are 2569, 4900, 5025 links?

Ans. 61 acres, 1 rood, 39 perches.

PROBLEM III.

To find the Area of a Trapezoid.

ADD together the two parallel sides; then multiply their sum by the perpendicular breadth, or the distance between them; and take half the product for the area. By Geom. theor. 29.

Ex. 1. In a trapezoid, the parallel sides are 750 and 1225, and the perpendicular distance between them 1540 links; to find the area.

1225
750
———
$1975 \times 770 = 1520750$ square links $=$ 15 acr. 33 perc.

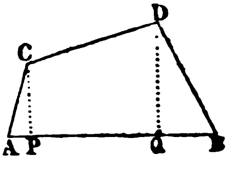
Ex. 2. How many square feet are contained in the plank, whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches?

Ans. $13\frac{13}{24}$ feet.

Ex. 3. In measuring along one side AB of a quadrangular field, that side, and the two perpendiculars let fall on it from the two opposite corners, measured as follow: required the content.

AP

AP = 110 links
AQ = 745
AB = 1110
CP = 352
DQ = 595



Ans. 4 acres, 1 rood, 5·792 perches.

PROBLEM IV.

To find the Area of any Trapezium.

DIVIDE the trapezium into two triangles by a diagonal; then find the areas of these triangles, and add them together.

Or thus, let fall two perpendiculars on the diagonal from the other two opposite angles; then add these two perpendiculars together, and multiply that sum by the diagonal, taking half the product for the area of the trapezium.

Ex. 1. To find the area of the trapezium, whose diagonal is 42, and the two perpendiculars on it 16 and 18.

Here $16 + 18 = 34$, its half is 17.
Then $42 \times 17 = 714$ the area.

Ex. 2. How many square yards of paving are in the trapezium, whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? Ans. $222\frac{1}{12}$ yards.

Ex. 3. In the quadrangular field ABCD, on account of obstructions there could only be taken the following measures, viz. the two sides BC 265 and AD 220 yards, the diagonal AC 378, and the two distances of the perpendiculars from the ends of the diagonal, namely, AE 100, and CF 70 yards. Required the construction of the figure, and the area in acres, when 4840 square yards make an acre?

Ans. 17 acres, 2 roods, 21 perches.

PROBLEM V.

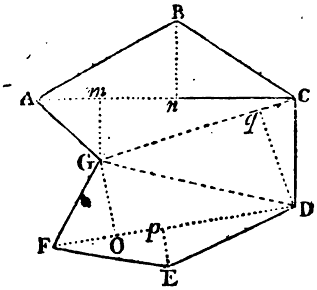
To find the Area of an Irregular Polygon.

DRAW diagonals dividing the proposed polygon into trapeziums and triangles. Then find the areas of all these separately, and add them together for the content of the whole polygon.

EXAMPLE.

EXAMPLE. To find the content of the irregular figure ABCDEFGA, in which are given the following diagonals and perpendiculars: namely,

AC	55
FD	52
GC	44
Gm	13
Bn	18
Go	12
Ep	8
Dq	23



Ans. $1878\frac{1}{2}$

PROBLEM VI.

To find the Area of a Regular Polygon.

RULE I. MULTIPLY the perimeter of the polygon, or sum of its sides, by the perpendicular drawn from its centre on one of its sides, and take half the product for the area *.

Ex. 1. To find the area of a regular pentagon, each side being 25 feet, and the perpendicular from the centre on each side 17·2047737.

Here $25 \times 5 = 125$ is the perimeter.
And $17{\cdot}2047737 \times 125 = 2150{\cdot}5967125$.
Its half 1075·298356 is the area sought.

RULE II. Square the side of the polygon; then multiply that square by the tabular area, or multiplier set against its name in the following table, and the product will be the area †.

No.

* This is only in effect resolving the polygon into as many equal triangles as it has sides, by drawing lines from the centre to all the angles; then finding their areas, and adding them all together.

† This rule is founded on the property, that like polygons, being similar figures, are to one another as the squares of their like sides; which is proved in the Geom. theor. 89. Now, the multipliers in the table, are the areas of the respective polygons to the side 1. Whence the rule is manifest.

Note.

No. of Sides.	Names.	Areas, or Multipliers.
3	Trigon or triangle	0·4330127
4	Tetragon or square	1·0000000
5	Pentagon	1·7204774
6	Hexagon	2·5980762
7	Heptagon	3·6339124
8	Octagon	4·8284271
9	Nonagon	6·1818242
10	Decagon	7·6942088
11	Undecagon	9·3656399
12	Dodecagon	11·1961524

EXAM. Taking here the same example as before, namely, a pentagon, whose side is 25 feet.

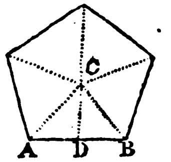
Then 25^2 being = to 625,
And the tabular area 1·7204774;
Theref. 1·7204774 × 625 = 1075·298375, as before.

Ex. 2. To find the area of the trigon or equilateral triangle, whose side is 20. Ans. 173·20508.

Ex. 3. To find the area of the hexagon whose side is 20. Ans. 1039·23048.

Ex. 4. To find the area of an octagon whose side is 20. Ans. 1931·37084.

Ex. 5. To find the area of a decagon whose side is 20. Ans. 3077·68352.



Note. The areas in the table, to each side 1, may be computed in the following manner: From the centre C of the polygon draw lines to every angle, dividing the whole figure into as many equal triangles as the polygon has sides; and let ABC be one of those triangles, the perpendicular of which is CD. Divide 360 degrees by the number of sides in the polygon, the quotient gives the angle at the centre ACB. The half of this gives the angle ACD; and this taken from 90°, leaves the angle CAD. Then it will be, as radius is to AD, so is tang. angle CAD, to the perpendicular CD. This perpendicular, multiplied by the half base AD, gives the area of the triangle ABC; which being multiplied by the number of the triangles, or of the sides of the polygon, gives its whole area, as in the table, for every one of the figures.

PROBLEM VII.

To find the Diameter and Circumference of any Circle, the one from the other.

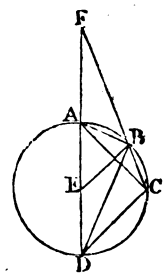
THIS may be done nearly by either of the two following proportions,

viz. As 7 is to 22, so is the diameter to the circumference.
Or, As 1 is to 3·1416, so is the diameter to the circumference *.

Ex. 1. To find the circumference of the circle whose diameter is 20.

By the first rule, as $7 : 22 :: 20 . 62\frac{6}{7}$, the answer.

Ex. 2.



* For, let ABCD be any circle, whose centre is E, and let AB, BC be any two equal arcs. Draw the several chords as in the figure, and join BE; also draw the diameter DA, which produce to F, till BF be equal to the chord BD.

Then the two isosceles triangles DEB, DBF, are equiangular, because they have the angle at D common; consequently $DE : DB :: DB : DF$. But the two triangles AFB, DCB are identical, or equal in all respects, because they have the angle F = the angle BDC, being each equal to the angle ADB, these being subtended by the equal arcs AB, BC; also the exterior angle FAB of the quadrangle ABCD, is equal to the opposite interior angle at C; and the two triangles have also the side BF = the side BD; therefore the side AF is also equal to the side DC. Hence the proportion above, viz. $DE : DB :: DB : DF = DA + AF$, becomes $DE : DB :: DB : 2DE + DC$. Then, by taking the rectangles of the extremes and means, it is $DB^2 = 2DE^2 + DE . DC$.

Now, if the radius DE be taken = 1, this expression becomes $DB^2 = 2 + DC$, and hence the root $DB = \sqrt{2 + DC}$. That is, if the measure of the supplemental chord of any arc be increased by the number 2, the square root of the sum will be the supplemental chord of half that arc.

Now, to apply this to the calculation of the circumference of the circle, let the arc AC be taken equal to $\frac{1}{6}$ of the circumference, and be successively bisected by the above theorem: thus, the chord AC of $\frac{1}{6}$ of the circumference, is the side of the inscribed regular hexagon, and is therefore equal to the radius AE or 1: hence, in the right-angled triangle ACD, it will be $DC = \sqrt{AD^2 - AC}$

Ex. 2. If the circumference of the earth be 25000 miles, what is its diameter?

By the 2d rule, as $3{\cdot}1416 : 1 :: 25000$ $7957\frac{1}{4}$ nearly the diameter.

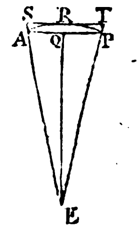
PROBLEM

$\sqrt{\text{AD}^2 - \text{AC}^2} = \sqrt{2^2 - 1^2} = \sqrt{3} = 1{\cdot}7320508076$, the supplemental chord of $\frac{1}{6}$ of the periphery.

Then, by the foregoing theorem, by always bisecting the arcs, and adding 2 to the last square root, there will be found the supplemental chords of the 12th, the 24th, the 48th, the 96th, &c, parts of the periphery; thus,

$\sqrt{3{\cdot}7320508076} =$	$1{\cdot}9318516525$	for the supplemental chord of	$\frac{1}{12}$ of the periphery.
$\sqrt{3{\cdot}9318516525} =$	$1{\cdot}9828897227$		$\frac{1}{24}$
$\sqrt{3{\cdot}9828897227} =$	$1{\cdot}9957178465$		$\frac{1}{48}$
$\sqrt{3{\cdot}9957178465} =$	$1{\cdot}9989291743$		$\frac{1}{96}$
$\sqrt{3{\cdot}9989291743} =$	$1{\cdot}9997322757$		$\frac{1}{192}$
$\sqrt{3{\cdot}9997322757} =$	$1{\cdot}9999330678$		$\frac{1}{384}$
$\sqrt{3{\cdot}9999330678} =$	$1{\cdot}9999832669$		$\frac{1}{768}$
$\sqrt{3{\cdot}9999832669} =$	- - - - -		$\frac{1}{1536}$

Since then it is found that $3{\cdot}9999832669$ is the square of the supplemental chord of the 1536th part of the periphery, let this number be taken from 4, which is the square of the diameter, and the remainder $0{\cdot}0000167331$ will be the square of the chord of the said 1536th part of the periphery, and consequently the root $\sqrt{0{\cdot}0000167331} = 0{\cdot}0040906112$ is the length of that chord; this number then being multiplied by 1536 gives $6{\cdot}2831788$ for the perimeter of a regular polygon of 1536 sides inscribed in the circle; which, as the sides of the polygon nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.



But now, to show how near this determination is to the truth, let AQP $= 0{\cdot}0040906112$ represent one side of such a regular polygon of 1536 sides, and SRT a side of another similar polygon described about the circle; and from the centre E let the perpendicular EQR be drawn, bisecting AP and ST in Q and R. Then since AQ is $= \frac{1}{2}$ AP $= 0{\cdot}0020453056$, and EA $= 1$, therefore $\text{EQ}^2 = \text{EA}^2 - \text{AQ}^2 = {\cdot}9999958167$, and consequently its root gives EQ $= {\cdot}9999979084$; then because of the parallels AP, ST, it is EQ : ER :: AP : ST :: as the whole inscribed perimeter : to the circumscribed one, that is, as $\cdot9999979084 : 1 :: 6{\cdot}2831788 : 6{\cdot}2831920$ the perimeter of the circumscribed polygon. Now, the circumference of the circle being greater than the

PROBLEM VIII.

To find the Length of any Arc of a Circle.

MULTIPLY the decimal ·01745 by the degrees in the given arc, and that product by the radius of the circle, for the length of the arc *.

Ex. 1. To find the length of an arc of 30 degrees, the radius being 9 feet. Ans. 4·7115.

Ex. 2. To find the length of an arc of $12^\circ\ 10'$, or $12^\circ\frac{1}{6}$, the radius being 10 feet. Ans. 2·1231.

PROBLEM IX.

To find the Area of a Circle †.

RULE I. Multiply half the circumference by half the diameter. Or multiply the whole circumference by the whole diameter, and take $\frac{1}{4}$ of the product.

RULE

---

the perimeter of the inner polygon, but less than that of the outer, it must consequently be greater than 6·2831788,
but less than 6·2831920,
and must therefore be nearly equal to $\frac{1}{2}$ their sum, or 6·2831854, which in fact is true to the last figure, which should be a 3, instead of the 4.

Hence the circumference being 6·2831854 when the diameter is 2, it will be the half of that, or 3·1415927, when the diameter is 1, to which the ratio in the rule, viz. 1 to 3·1416 is very near. Also the other ratio in the rule, 7 to 22 or 1 to $3\frac{1}{7} = 3·1428$ &c, is another near approximation.

* It having been found, in the demonstration of the foregoing problem, that when the radius of a circle is 1, the length of the whole circumference is 6·2831854, which consists of 360 degrees; therefore as $360^\circ : 6·2831854 :: 1^\circ : ·01745$ &c, the length of the arc of 1 degree. Hence the decimal ·01745 multiplied by any number of degrees, will give the length of the arc of those degrees. And because the circumferences and arcs are in proportion as the diameters, or as the radii of the circles, therefore as the radius 1 is to any other radius r, so is the length of the arc above mentioned, to $·01745 \times$ degrees in the arc $\times\ r$, which is the length of that arc, as in the rule.

† The first rule is proved in the Geom. theor. 94.

And the 2d and 3d rules are deduced from the first rule, in this manner.—By that rule, $dc \div 4$ is the area, when d denotes the

Rule II. Square the diameter, and multiply that square by the decimal $\cdot 7854$, for the area.

Rule III. Square the circumference, and multiply that square by the decimal $\cdot 07958$.

Ex. 1. To find the area of a circle whose diameter is 10, and its circumference 31·416.

By Rule 1.	By Rule 2.	By Rule 3.
31·416	·7854	31·416
10	$10^2 =$ 100	31·416
4) 314·16	78·54	986·965
78·54		·07958
		78·54

So that the area is 78·54 by all the three rules.

Ex. 2. To find the area of a circle, whose diameter is 7, and circumference 22. Ans. $38\frac{1}{2}$.

Ex. 3. How many square yards are in a circle whose diameter is $3\frac{1}{2}$ feet? Ans. 1·069

Ex. 4. To find the area of a circle whose circumference is 12 feet. Ans. 11·4595.

PROBLEM X.

To find the Area of a Circular Ring, or of the Space included between the Circumferences of two Circles; the one being contained within the other.

Take the difference between the areas of the two circles, as found by the last problem, for the area of the ring.—Or,

---

the diameter, and c the circumference. But, by prob. 7, c is $= 3\cdot1416\,d$; therefore the said area $dc \div 4$, becomes $d \times 3\cdot1416\,d \div 4 = \cdot7854\,d^2$, which gives the 2d rule.—Also by the same prob. 7, d is $= c \div 3\cdot1416$; therefore again the same first area $dc \div 4$, becomes $c \div 3\cdot1416 \times c \div 4 = c^2 \div 12\cdot5664$, which is $= c^2 \times \cdot07958$, by taking the reciprocal of $12\cdot5664$, or changing that divisor into the multiplier $\cdot07958$; which gives the 3d rule.

Corol. Hence the areas of different circles are in proportion to one another, as the square of their diameters or as the square of their circumferences; as before proved in the Geom. theor. 93.

which

which is the same thing, subtract the square of the less diameter from the square of the greater, and multiply their difference by ·7854.—Or lastly, multiply the sum of the diameters by the difference of the same, and that product by ·7854; which is still the same thing, because the product of the sum and difference of any two quantities, is equal to the difference of their squares.

Ex. 1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences.

Here $10 + 6 = 16$ the sum, and $10 - 6 = 4$ the diff.
Therefore $·7854 \times 16 \times 4 = ·7854 \times 64 = 50·2656$, the area.

Ex. 2. What is the area of the ring, the diameters of whose bounding circles are 10 and 20? Ans. 235·62.

PROBLEM XI.

To find the Area of the Sector of a Circle.

RULE I. MULTIPLY the radius, or half the diameter, by half the arc of the sector, for the area. Or, multiply the whole diameter by the whole arc of the sector, and take $\frac{1}{4}$ of the product. The reason of which is the same as for the first rule to problem 9, for the whole circle.

RULE II. Compute the area of the whole circle: then say, as 360 is to the degrees in the arc of the sector, so is the area of the whole circle, to the area of the sector.

This is evident, because the sector is proportional to the length of the arc, or to the degrees contained in it.

Ex. 1. To find the area of a circular sector, whose arc contains 18 degrees; the diameter being 3 feet?

1. By the 1st Rule.

First, $3·1416 \times 3 = 9·4248$, the circumference.
And $360 : 18 :: 9·4248 :: ·47124$, the length of the arc.
Then $·47124 \times 3 \div 4 = 1·41372 \div 4 = ·35343$, the area.

2. By the 2d Rule.

First, $·7854 \times 3^2 = 7·0686$, the area of the whole circle.
Then, as $360 : 18 :: 7·0686 : ·35343$, the area of the sector.

Ex. 2.

Ex. 2. To find the area of a sector, whose radius is 10, and arc 20. Ans. 100.

Ex. 3. Required the area of a sector, whose radius is 25, and its arc containing $147^\circ\ 29'$. Ans. 804·3986.

PROBLEM XII.

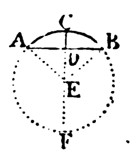
To find the Area of a Segment of a Circle.

RULE I. FIND the area of the sector having the same arc with the segment, by the last problem.

Find also the area of the triangle, formed by the chord of the segment and the two radii of the sector.

Then add these two together for the answer, when the segment is greater than a semicircle: or subtract them when it is less than a semicircle.—As is evident by inspection.

Ex. 1. To find the area of the segment ACBDA, its chord AB being 12, and the radius AE or CE 10.



First, As AE : sin. $\angle$ D 90° :: AD : sin. $36^\circ\ 52'\frac{1}{5} = 36{\cdot}87$ degrees, the degrees in the $\angle$ AEC or arc AC. Their double, 73·74, are the degrees in the whole arc ACB.

Now $\cdot 7854 \times 400 = 314{\cdot}16$, the area of the whole circle.

Therefore $360^\circ : 73{\cdot}74 :: 314{\cdot}16 : 64{\cdot}3504$, area of the sector ACBE.

Again, $\sqrt{AE^2 - AD^2} = \sqrt{100 - 36} = \sqrt{64} = 8 =$ DE.

Theref. AD $\times$ DE $= 6 \times 8 = 48$, the area of the triangle AEB.

Hence sector ACBE $-$ triangle AEB $= 16{\cdot}3504$, area of seg. ACBDA.

RULE II. Divide the height of the segment by the diameter, and find the quotient in the column of heights in the following tablet: Take out the corresponding area in the next column on the right hand; and multiply it by the square of the circle's diameter, for the area of the segment *.

Note.

* The truth of this rule depends on the principle of similar plane figures, which are to one another as the square of their like linear dimensions. The segments in the table are those of a circle

Ex. 2. The length of an irregular figure being 84, and the breadths at six equidistant places 17·4, 20·6, 14·2, 16·5, 20·1, 24·4; what is the area? Ans. 1550·64.

PROBLEM XIV.

To find the Area of an Ellipsis or Oval.

MULTIPLY the longest diameter, or axis, by the shortest; then multiply the product by the decimal ·7854, for the area. As appears from cor. 2, theor. 3, of the Ellipse, in the Conic Sections.

Ex. 1. Required the area of an ellipse whose two axes are 70 and 50. Ans. 2748·9.

Ex. 2. To find the area of the oval whose two axes are 24 and 18. Ans. 339·2928.

PROBLEM XV.

To find the Area of an Elliptic Segment.

FIND the area of a corresponding circular segment, having the same height and the same vertical axis or diameter. Then say, as the said vertical axis is to the other axis, parallel to the segment's base, so is the area of the circular segment before found, to the area of the elliptic segment sought. This rule also comes from cor. 2, theor. 3, of the Ellipse.

Otherwise thus. Divide the height of the segment by the vertical axis of the ellipse; and find, in the table of circular segments to prob. 12, the circular segment having the above quotient for its versed sine: then multiply all together, this segment and the two axes of the ellipse, for the area.

Ex. 1. To find the area of the elliptic segment, whose height is 20, the vertical axis being 70, and the parallel axis 50.

---

which is the whole area, agreeing with the rule: *m* being the arithmetical mean between the extremes, or half the sum of them both, and 4 the number of the parts. And the same for any other number of parts whatever.

Here

Here 20 ÷ 70 gives $\cdot28\frac{4}{7}$ the quotient or versed sine; to which in the table answers the seg. ·18518

then 70

12·96260

50

648·13000 the area.

Ex. 2. Required the area of an elliptic segment, cut off parallel to the shorter axis; the height being 10, and the two axes 25 and 35. – Ans. 162·03.

Ex. 3. To find the area of the elliptic segment, cut off parallel to the longer axis; the height being 5, and the axes 25 and 35. Ans. 97·8425.

PROBLEM XVI.

To find the Area of a Parabola, or its Segment.

MULTIPLY the base by the perpendicular height; then take two-thirds of the product for the area. As is proved in theorem 17 of the Parabola, in the Conic Sections.

Ex. 1. To find the area of a parabola; the height being 2, and the base 12.

Here $2 \times 12 = 24$. Then $\frac{2}{3}$ of $24 = 16$, is the area.

Ex. 2. Required the area of the parabola, whose height is 10, and its base 16. Ans. $106\frac{2}{3}$.

MENSURATION OF SOLIDS.

BY the Mensuration of Solids are determined the spaces included by contiguous surfaces; and the sum of the measures of these including surfaces, is the whole surface or superficies of the body.

The measure of a solid, is called its solidity, capacity, or content.

Solids are measured by cubes, whose sides are inches, or feet, or yards, &c. And hence the solidity of a body is said to be so many cubic inches, feet, yards, &c, as will fill its capacity or space, or another of an equal magnitude.

Th

The least solid measure is the cubic inch, other cubes being taken from it according to the proportion in the following table, which is formed by cubing the linear proportions.

Table of Cubic or Solid Measures.

1728	cubic inches make		1 cubic foot
27	cubic feet	-	1 cubic yard
$166\frac{3}{8}$	cubic yards	-	1 cubic pole
64000	cubic poles	-	1 cubic furlong
512	cubic furlongs	-	1 cubic mile.

PROBLEM I.

To find the Superficies of a Prism or Cylinder.

MULTIPLY the perimeter of one end of the prism by the length or height of the solid, and the product will be the surface of all its sides. To which add also the area of the two ends of the prism, when required *.

Or, compute the areas of all the sides and ends separately, and add them all together.

Ex. 1. To find the surface of a cube, the length of each side being 20 feet. Ans. 2400 feet.

Ex. 2. To find the whole surface of a triangular prism, whose length is 20 feet, and each side of its end or base 18 inches. Ans. 91·948 feet.

Ex. 3. To find the convex surface of a round prism, or cylinder, whose length is 20 feet, and the diameter of its base is 2 feet. Ans. 125·664.

Ex. 4. What must be paid for lining a rectangular cistern with lead, at 3*d.* a pound weight, the thickness of the lead being such as to weigh 7lb. for each square foot of surface; the inside dimensions of the cistern being as follow, viz. the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches? Ans. 3*l.* 5*s.* $9\frac{3}{4}$*d.*

---

* The truth of this will easily appear, by considering that the sides of any prism are parallelograms, whose common length is the same as the length of the solid, and their breadths taken all together make up the perimeter of the ends of the same.

And the rule is evidently the same for the surface of a cylinder.

PROBLEM

PROBLEM II.

To find the Surface of a Pyramid or Cone.

MULTIPLY the perimeter of the base by the slant height, or length of the side, and half the product will evidently be the surface of the sides, or the sum of the areas of all the triangles which form it. To which add the area of the end or base, if requisite.

Ex. 1. What is the upright surface of a triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet? Ans. 90 feet.

Ex. 2. Required the convex surface of a cone, or circular pyramid, the slant height being 50 feet, and the diameter of its base $8\frac{1}{2}$ feet. Ans. 667·59.

PROBLEM III.

To find the Surface of the Frustum of a Pyramid or Cone, being the lower part, when the top is cut off by a plane parallel to the base.

ADD together the perimeters of the two ends, and multiply their sum by the slant height, taking half the product for the answer.—As is evident, because the sides of the solid are trapezoids, having the opposite sides parallel.

Ex. 1. How many square feet are in the surface of the frustum of a square pyramid, whose slant height is 10 feet; also each side of the base or greater end being 3 feet 4 inches, and each side of the less end 2 feet 2 inches? Ans. 110 feet.

Ex. 2. To find the convex surface of the frustum of a cone, the slant height of the frustum being $12\frac{1}{2}$ feet, and the circumferences of the two ends 6 and 8·4 feet. Ans. 90 feet.

PROBLEM IV.

To find the Solid Content of any Prism or Cylinder.

Find the area of the base, or end, whatever the figure of it may be; and multiply it by the length of the prism or cylinder, for the solid content *.

Note.

* This rule appears from the Geom. theor. 110, cor. 2. The same is more particularly shown as follows: Let the annexed rectangular

Note. For a cube, take the cube of its side by multiplying this twice by itself; and for a parallelopipedon, multiply the length, breadth, and depth all together, for the content.

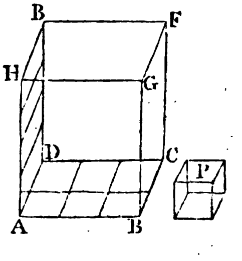
Ex. 1. To find the solid content of a cube, whose side is 24 inches. Ans. 13824.

Ex. 2. How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet 8 inches, and thickness 2 feet 6 inches? Ans. $21\frac{1}{9}$.

Ex. 3. How many gallons of water will the cistern contain, whose dimensions are the same as in the last example, when 282 cubic inches are contained in one gallon? Ans. $129\frac{17}{47}$.

Ex. 4. Required the solidity of a triangular prism, whose length is 10 feet, and the three sides of its triangular end or base are 3, 4, 5 feet. Ans. 60.

Ex. 5. Required the content of a round pillar, or cylinder, whose length is 20 feet, and circumference 5 feet 6 inches. Ans. 48·1459 feet.



rectangular parallelopipedon be the solid to be measured, and the cube P the solid measuring unit, its side being 1 inch, or 1 foot, &c; also, let the length and breadth of the base ABCD, and also the height AH, be each divided into spaces equal to the length of the base of the cube P, namely, here 3 in the length and 2 in the breadth, making 3 times 2 or 6 squares in the base AC, each equal to the base of the cube P. Hence it is manifest that the parallelopipedon will contain the cube P, as many times as the base AC contains the base of the cube, repeated as often as the height AH contains the height of the cube. That is, the content of any parallelopipedon is found, by multiplying the area of the base by the altitude of that solid.

And because all prisms and cylinders are equal to parallelopipedons of equal bases and altitudes, by Geom. theor. 103, it follows that the rule is general for all such solids, whatever the figure of the base may be.

PROBLEM

PROBLEM V.

To find the Content of any Pyramid or Cone.

FIND the area of the base, and multiply that area by the perpendicular height; then take $\frac{1}{3}$ of the product for the content *.

Ex. 1. Required the solidity of a square pyramid, each side of its base being 30, and its perpendicular height 25.
Ans. 7500.

Ex. 2. To find the content of a triangular pyramid, whose perpendicular height is 30, and each side of the base 3.
Ans. 38·97117.

Ex. 3. To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three sides of its base 5, 6, 7 feet. Ans. 71·0352.

Ex. 4. What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet?
Ans. 27·5276.

Ex. 5. What is the content of the hexagonal pyramid, whose height is 6·4 feet, and each side of its base 6 inches?
Ans. 1·38564 feet.

Ex. 6. Required the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its base 9 feet.
Ans. 22·56093.

PROBLEM VI.

To find the Solidity of the Frustum of a Cone or Pyramid.

ADD into one sum, the areas of the two ends, and the mean proportional between them: and take $\frac{1}{3}$ of that sum for a mean area; which being multiplied by the perpendicular height, or length of the frustum, will give its content †.
Note.

---

* This rule follows from that of the prism, because any pyramid is $\frac{1}{3}$ of a prism of equal base and altitude; by Geom. theor. 115, cor. 1 and 2.

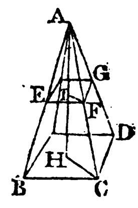
† Let ABCD be any pyramid, of which BCDGFE is a frustum. And put a^2 for the area of the base BCD, b^2 the area of the top
EFG

Note. This general rule may be otherwise expressed, as follows, when the ends of the frustum are circles or regular polygons. In this latter case, square one side of each polygon, and also multiply the one side by the other; add all these three products together; then multiply their sum by the tabular area proper to the polygon, and take one-third of the product for the mean area, to be multiplied by the length, to give the solid content. And in the case of the frustum of a cone, the ends being circles, square the diameter or the circumference of each end, and also multiply the same two dimensions together; then take the sum of the three products, and multiply it by the proper tabular number, viz. by ·7854 when the diameters are used, or by ·07958 in using the circumferences; then taking one-third of the product, to multiply by the length, for the content.

Ex. 1. To find the number of solid feet in a piece of timber, whose bases are squares, each side of the greater end being 15 inches, and each side of the less end 6 inches; also, the length or the perpendicular altitude 24 feet. Ans. $19\frac{1}{2}$.

Ex. 2. Required the content of a pentagonal frustum, whose height is 5 feet, each side of the base 18 inches; and each side of the top or less end 6 inches. Ans. 9·31925 feet.

EFG, h the height IH of the frustum, and c the height AI of the top part above it. Then $c+h =$ AH is the height of the whole pyramid.



Hence, by the last prob. $\frac{1}{3}a^2(c+h)$ is the content of the whole pyramid ABCD, and $\frac{1}{3}b^2c$ the content of the top part AEFG; therefore the difference $\frac{1}{3}a^2(c+h) - \frac{1}{3}b^2c$ is the content of the frustum BCDGFE. But the quantity c being no dimension of the frustum, it must be expelled from this formula, by substituting its value, found in the following manner. By Geom. theor. 112, $a^2 : b^2 :: (c+h)^2 : c^2$, or $a : b :: c+h : c$, hence (Geom. th. 69) $a-b : b :: h : c$, and $a-b : a :: h : c+h$; hence therefore $c = \frac{bh}{a-b}$, and $c+h = \frac{ah}{a-b}$; then these values of c and $c+h$ being substituted for them in the expression for the content of the frustum, gives that content $= \frac{1}{3}a^2 \times \frac{ah}{a-b} - \frac{1}{3}b^2 \times \frac{bh}{a-b} = \frac{1}{3}h \times \frac{a^3-b^3}{a-b} = \frac{1}{3}h \times (a^2 + ab+b^2)$; which is the rule above given; ab being the mean between a^2 and b^2.

Ex. 3.

Ex. 3. To find the content of a conic frustum, the altitude being 18, the greatest diameter 8, and the least diameter 4. Ans. 527·7888.

Ex. 4. What is the solidity of the frustum of a cone, the altitude being 25, also the circumference at the greater end being 20, and at the less end 10? Ans. 464·216.

Ex. 5. If a cask, which is two equal conic frustums joined together at the bases, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold? Ans. 79·0613.

PROBLEM VII.

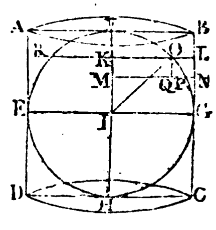
To find the Surface of a Sphere, or any Segment.

RULE I. MULTIPLY the circumference of the sphere by its diameter, and the product will be the whole surface of it *.

RULE II.

---

* These rules come from the following theorems for the surface of a sphere, viz. That the said surface is equal to the curve surface of its circumscribing cylinder; or that it is equal to 4 great circles of the same sphere, or of the same diameter: which are thus proved.



Let ABCD be a cylinder, circumscribing the sphere EFGH; the former generated by the rotation of the rectangle FBCH about the axis or diameter FH; and the latter by the rotation of the semicircle FGH about the same diameter FH. Draw two lines KL, MN, perpendicular to the axis, intercepting the parts LN, OP, of the cylinder and sphere; then will the ring or cylindric surface generated by the rotation of LN, be equal to the ring or spherical surface generated by the arc OP. For first, suppose the parallels KL and MN to be indefinitely near together; drawing IO, and also OQ parallel to LN. Then, the two triangles IKO, OQP, being equiangular, it is, as OP : OQ or LN : : IO or KL : KO : : circumference described by KL : circumf. described by KO; therefore the rectangle OP × circumf. of KO is equal to the rectangle LN × circumf. of KL; that is, the ring described by OP on the sphere, is equal to the ring described by LN on the cylinder.

Rule II. Square the diameter and multiply that square by 3·1416, for the surface.

Rule III. Square the circumference; then either multiply that square by the decimal ·3183, or divide it by 3·1416, for the surface.

Note. For the surface of a segment or frustum, multiply the whole circumference of the sphere by the height of the part required.

Ex. 1. Required the convex superficies of a sphere, whose diameter is 7, and circumference 22. Ans. 154.

Ex. 2. Required the superficies of a globe, whose diameter is 24 inches. Ans. 1809·5616.

Ex. 3. Required the area of the whole surface of the earth, its diameter being $7957\frac{3}{4}$ miles, and its circumference 25000 miles. Ans. 198943750 sq. miles.

Ex. 4. The axis of a sphere being 42 inches, what is the convex superficies of the segment whose height is 9 inches? Ans. 1187·5248 inches.

Ex. 5. Required the convex surface of a spherical zone, whose breadth or height is 2 feet, and cut from a sphere of $12\frac{1}{2}$ feet diameter. Ans. 78·54 feet.

And as this is every where the case, therefore the sums of any corresponding number of these are also equal; that is, the whole surface of the sphere, described by the whole semicircle FGH, is equal to the whole curve surface of the cylinder, described by the height BC; as well as the surface of any segment described by FO, equal to the surface of the corresponding segment described by BL.

Corol. 1. Hence the surface of the sphere is equal to 4 of its great circles, or equal to the circumference EFGH, or of DC, multiplied by the height BC, or by the diameter FH.

Corol. 2. Hence also, the surface of any such part as a segment or frustum, or zone, is equal to the same circumference of the sphere, multiplied by the height of the said part. And consequently such spherical curve surfaces are to one another in the same proportion as their altitudes.

PROBLEM

PROBLEM VIII.

To find the Solidity of a Sphere or Globe.

RULE I. Multiply the surface by the diameter, and take $\frac{1}{6}$ of the product for the content*. Or, which is the same thing, multiply the square of the diameter by the circumference, and take $\frac{1}{6}$ of the product.

RULE II. Take the cube of the diameter, and multiply it by the decimal ·5236, for the content.

RULE III. Cube the circumference, and multiply by ·01688.

Ex. 1. To find the content of a sphere whose axis is 12.
Ans. 904·7808.

Ex. 2. To find the solid content of the globe of the earth, supposing its circumference to be 25000 miles.
Ans. 263750000000 miles.

PROBLEM IX.

To find the Solid Content of a Spherical Segment.

† RULE I. From 3 times the diameter of the sphere take

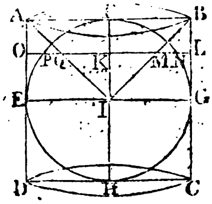
---

* For, put d = the diameter, c = the circumference, and s = the surface of the sphere, or of its circumscribing cylinder; also, a = the number 3·1416.

Then, $\frac{1}{4}s$ is = the base of the cylinder, or one great circle of the sphere; and d is the height of the cylinder; therefore $\frac{1}{4}ds$ is the content of the cylinder. But $\frac{2}{3}$ of the cylinder is the sphere, by th. 117, Geom. that is, $\frac{2}{3}$ of $\frac{1}{4}ds$, or $\frac{1}{6}ds$ is the sphere; which is the first rule.

Again, because the surface s is $= ad^2$; therefore $\frac{1}{6}ds = \frac{1}{6}ad^3 = \cdot5236d^3$, is the content, as in the 2d rule. Also, d being $= c \div a$, therefore $\frac{1}{6}ad^3 = \frac{1}{6}c^3 \div a^2 = \cdot01688$, the 3d rule for the content.

† By corol. 3, of theor. 117, Geom. it appears that the spheric segment PFN, is equal to the difference between the cylinder ABLO, and the conic frustum ABMQ.



But, putting d = AB or FH the diameter of the sphere or cylinder, h = FK the height of the segment, r = PK the radius of its base, and a = 3·1416; then the content of the cone ABI is $= \frac{1}{4}ad^2 \times \frac{1}{3}$FI $= \frac{1}{24}ad^3$; and by the similar cones ABI, QMI, as

 FI²

take double the height of the segment; then multiply the remainder by the square of the height, and the product by the decimal ·5236, for the content.

RULE II. To 3 times the square of the radius of the segment's base, add the square of its height; then multiply the sum by the height, and the product by ·5236, for the content.

Ex. 1. To find the content of a spherical segment, of 2 feet in height, cut from a sphere of 8 feet diameter.

Ans. 41·888.

Ex. 2. What is the solidity of the segment of a sphere, its height being 9, and the diameter of its base 20?

Ans. 1795·4244.

Note. The general rules for measuring all sorts of figures having been now delivered, we may next proceed to apply them to the several practical uses in life, as follows.

$FI^3 : KI^3 :: \frac{1}{24}ad^3 : \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d - h}{\frac{1}{2}d})^3 =$ the cone QMI; therefore the cone ABI — the cone QMI $= \frac{1}{24}ad^3 - \frac{1}{24}ad^3 \times (\frac{\frac{1}{2}d - h}{\frac{1}{2}d})^3 = \frac{1}{4}ad^2h - \frac{1}{2}adh^2 + \frac{1}{3}ah^3$ is $=$ the conic frustum ABMQ.

And $\frac{1}{4}ad^2h$ is $=$ the cylinder ABLO.

Then the difference of these two is $\frac{1}{2}adh^2 - \frac{1}{3}ah^3 = \frac{1}{6}ah^2 \times (3d - 2h)$, for the spheric segment PFN; which is the first rule.

Again, because $PK^2 = FK \times KH$ (cor. to theor. 87, Geom.) or $r^2 = h\ (d - h)$, therefore $d = \frac{r^2}{h} + h$, and $3d - 2h = \frac{3r^2}{h} + h = \frac{3r^2 + h^2}{h}$; which being substituted in the former rule, it becomes $\frac{1}{6}ah^2 \times \frac{3r^2 + h^2}{h} = \frac{1}{6}ah \times (3r^2 + h^2)$, which is the 2d rule.

Note. By subtracting a segment from a half sphere, or from another segment, the content of any frustum or zone may be found.

LAND

LAND SURVEYING.

SECTION I.

DESCRIPTION AND USE OF THE INSTRUMENTS.

1. OF THE CHAIN.

LAND is measured with a chain, called Gunter's Chain, from its inventor, the length of which is 4 poles, or 22 yards, or 66 feet. It consists of 100 equal links; and the length of each link is therefore $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches.

Land is estimated in acres, roods, and perches. An acre is equal to 10 square chains, or as much as 10 chains in length and 1 chain in breadth. Or, in yards, it is $220 \times 22 = 4840$ square yards. Or, in poles, it is $40 \times 4 = 160$ square poles. Or, in links, it is $1000 \times 100 = 100000$ square links: these being all the same quantity.

Also, an acre is divided into 4 parts called roods, and a rood into 40 parts called perches, which are square poles, or the square of a pole of $5\frac{1}{2}$ yards long, or the square of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 square links. So that the divisions of land measure, will be thus:

625 sq. links = 1 pole or perch
40 perches = 1 rood
4 roods = 1 acre.

The length of lines, measured with a chain, are best set down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore, after the content is found, it will be in square links; then cut off five of the figures on the right-hand for decimals, and the rest will be acres. These decimals are then multiplied by 4 for roods, and the decimals of these again by 40 for perches.

EXAM. Suppose the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

$$\begin{array}{r} 792 \\ 385 \\ \hline 3960 \\ 6336 \\ 2376 \\ \hline 3{\cdot}04920 \end{array} \qquad \begin{array}{r} 3{\cdot}04920 \\ 4 \\ \hline {\cdot}19680 \\ 40 \\ \hline 7{\cdot}87200 \end{array}$$

Ans. 3 acres, 0 roods, 7 perches.

2. OF THE PLAIN TABLE.

THIS instrument consists of a plain rectangular board, of any convenient size: the centre of which, when used, is fixed by means of screws to a three-legged stand, having a ball and socket, or other joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belong various parts, as follow.

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a sheet of paper on the table. One side of this frame is usually divided into equal parts, for drawing lines across the table, parallel or perpendicular to the sides; and the other side of the frame is divided into 360 degrees, to a centre in the middle of the table; by means of which the table may be used as a theodolite, &c.

2. A magnetic needle and compass, either screwed into the side of the table, or fixed beneath its centre, to point out the directions, and to be a check on the sights.

3. An index, which is a brass two-foot scale, with either a small telescope, or open sights set perpendicularly on the ends. These sights and one edge of the index are in the same plane, and that is called the fiducial edge of the index.

To use this instrument, take a sheet of paper which will cover it, and wet it to make it expand; then spread it flat on the table, pressing down the frame on the edges, to stretch it and keep it fixed there; and when the paper is become dry, it will, by contracting again, stretch itself smooth and flat from any cramps and unevenness. On this paper is to be drawn the plan or form of the thing measured.

Thus, begin at any proper part of the ground, and make a point on a convenient part of the paper or table, to represent that place on the ground; then fix in that point one leg of the compasses, or a fine steel pin, and apply to it the fiducial edge of the index, moving it round till through the sights you perceive some remarkable object, as the corner of a field, &c; and from the station-point draw a line with the point of the compasses along the fiducial edge of the index, which is called setting or taking the object: then set another object or corner, and draw its line; do the same by another; and so on, till as many objects are taken as may be thought fit. Then measure from the station towards as many of the objects as may be necessary, but not more, taking the requisite offsets to corners or crooks in the hedges, laying the measures down on their respective lines on the table.

Then

Then at any convenient place measured to, fix the table in the same position, and set the objects which appear from that place; and so on, as before. And thus continue till the work is finished, measuring such lines only as are necessary, and determining as many as may be by intersecting lines of direction drawn from different stations.

Of shifting the Paper on the Plain Table.

When one paper is full, and there is occasion for more; draw a line in any manner through the farthest point of the last station line, to which the work can be conveniently laid down; then take the sheet off the table, and fix another on, drawing a line over it, in a part the most convenient for the rest of the work; then fold or cut the old sheet by the line drawn on it, applying the edge to the line on the new sheet, and, as they lie in that position, continue the last station line on the new paper, placing on it the rest of the measure, beginning at where the old sheet left off. And so on from sheet to sheet.

When the work is done, and you would fasten all the sheets together into one piece, or rough plan, the aforesaid lines are to be accurately joined together, in the same manner as when the lines were transferred from the old sheets to the new ones. But it is to be noted, that if the said joining lines, on the old and new sheets, have not the same inclination to the side of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to respect still the same degree of the compass, the easiest way of drawing the line in the same position, is to draw them both parallel to the same sides of the table, by means of the equal divisions marked on the other two sides.

3. OF THE THEODOLITE.

The theodolite is a brazen circular ring, divided into 360 degrees, &c, and having an index with sights, or a telescope, placed on the centre, about which the index is moveable; also a compass fixed to the centre, to point out courses and check the sights; the whole being fixed by the centre on a stand of a convenient height for use.

In using this instrument, an exact account, or field-book, of all measures and things necessary to be remarked in the plan, must be kept, from which to make out the plan on returning home from the ground.

Begin at such part of the ground, and measure in such directions as are judged most convenient; taking angles or directions to objects, and measuring such distances as appear

necessar

necessary, under the same restrictions as in the use of the plain table. And it is safest to fix the theodolite in the original position at every station, by means of fore and back objects, and the compass, exactly as in using the plain table; registering the number of degrees cut off by the index when directed to each object; and, at any station, placing the index at the same degree as when the direction towards that station was taken from the last preceding one, to fix the theodolite there in the original position.

The best method of laying down the aforesaid lines of direction, is to describe a pretty large circle; then quarter it, and lay on it the several numbers of degrees cut off by the index in each direction, and drawing lines from the centre to all these marked points in the circle. Then, by means of a parallel ruler, draw from station to station, lines parallel to the aforesaid lines drawn from the centre to the respective points in the circumference.

4. OF THE CROSS.

THE cross consists of two pair of sights set at right angles to each other, on a staff having a sharp point at the bottom, to fix in the ground.

The cross is very useful to measure small and crooked pieces of ground. The method is, to measure a base or chief line, usually in the longest direction of the piece, from corner to corner; and while measuring it, finding the places where perpendiculars would fall on this line, from the several corners and bends in the boundary of the piece, with the cross, by fixing it, by trials, on such parts of the line, as that through one pair of the sights both ends of the line may appear, and through the other pair the corresponding bends or corners; and then measuring the lengths of the said perpendiculars.

REMARKS.

Besides the fore-mentioned instruments, which are most commonly used, there are some others; as,

The perambulator, used for measuring roads, and other great distances, level ground, and by the sides of rivers. It has a wheel of $8\frac{1}{4}$ feet, or half a pole, in circumference, by the turning of which the machine goes forward: and the distance measured is pointed out by an index, which is moved round by clock work.

Levels, with telescopic or other sights, are used to find the level between place and place, or how much one place is higher or lower than another. And in measuring any sloping or oblique line, either ascending or descending, a small pocket

pocket level is useful for showing how many links for each chain are to be deducted, to reduce the line to the horizontal length.

An offset-staff is a very useful instrument, for measuring the offsets and other short distances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten small arrows, or rods of iron or wood, are used to mark the end of every chain length, in measuring lines. And sometimes pickets, or staves with flags, are set up as marks or objects of direction.

Various scales are also used in protracting and measuring on the plan or paper; such as plane scales, line or chords, protractor, compasses, reducing scale, parallel and perpendicular rules, &c. Of plane scales, there should be several sizes, as a chain in 1 inch, a chain in $\frac{3}{4}$ of an inch, a chain in $\frac{1}{2}$ an inch, &c. And of these, the best for use are those that are laid on the very edges of the ivory scale, to mark off distances, without compasses.

SECTION II.

THE PRACTICE OF SURVEYING.

THIS part contains the several works proper to be done in the field, or the ways of measuring by all the instruments, and in all situations.

PROBLEM I.

To Measure a Line or Distance.

To measure a line on the ground with the chain: Having provided a chain, with 10 small arrows, or rods, to fix one into the ground, as a mark, at the end of every chain; two persons take hold of the chain, one at each end of it; and all the 10 arrows are taken by one of them, who goes foremost, and is called the leader; the other being called the follower, for distinction's sake.

A picket, or station-staff, being set up in the direction of the line to be measured, if there do not appear some marks naturally in that direction, they measure straight towards it, the leader fixing down an arrow at the end of every chain, which the follower always takes up, as he comes at it, till all the ten arrows are used. They are then all returned to the leader, to use over again. And thus the arrows are changed from the one to the other at every 10 chains length, till the whole line is finished; then the number of chang

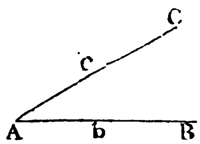
of the arrows shows the number of tens, to which the follower adds the arrows he holds in his hand, and the number of links of another chain over to the mark or end of the line. So, if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is set down in links thus, 3645.

When the ground is not level, but either ascending or descending; at every chain length, lay the offset-staff, or link-staff, down in the slope of the chain, on which lay the small pocket level, to show how many links or parts the slope line is longer than the true level one; then draw the chain forward so many links or parts, which reduces the line to the horizontal direction.

PROBLEM II.

To take Angles and Bearings.

Let B and C be two objects, or two pickets set up perpendicular; and let it be required to take their bearings, or the angles formed between them at any station A.



1. *With the Plain Table.*

The table being covered with a paper, and fixed on its stand; plant it at the station A, and fix a fine pin, or a foot of the compasses, in a proper point of the paper, to represent the place A: Close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights: then by the fiducial edge of the index draw a line. In the same manner draw another line in the direction of the other object C. And it is done.

2. *With the Theodolite, &c.*

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till the mark B is seen through these sights; and there screw the instrument fast. Then turn the moveable index round, till through its sights the other mark C is seen. Then the degrees cut by the index, on the graduated limb or ring of the instrument, show the *quantity* of the angle.

3. *With*

3. *With the Magnetic Needle and Compass.*

Turn the instrument or compass so, that the north end of the needle point to the flower-de-luce. Then direct the sights to one mark as B, and note the degrees cut by the needle. Next direct the sights to the other mark C, and note again the degrees cut by the needle. Then their sum or difference, as the case may be, will give the quantity of the angle BAC.

4. *By Measurement with the Chain, &c.*

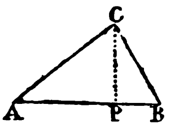
Measure one chain length, or any other length, along both directions, as to b and c. Then measure the distance b c, and it is done.—This is easily transferred to paper, by making a triangle Abc with these three lengths, and then measuring the angle A.

PROBLEM III.

To Survey a Triangular Field ABC.

1. *By the Chain.*

AP	794
AB	1321
PC	826



Having set up marks at the corners, which is to be done in all cases where there are not marks naturally; measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting down the distance AP. Then complete the distance AB, by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area from them is easily found. Or having the place P of the perpendicular, the triangle is easily constructed.

Or, measure all the three sides with the chain, and note them down. From which the content is easily found, or the figure is constructed.

2. *By taking some of the Angles.*

Measure two sides AB, AC, and the angle A between them. Or measure one side AB, and the two adjacent angles A and B. From either of these ways the figure is easily planned; then by measuring the perpendicular CP on the plan, and multiplying it by half AB, the content is found.

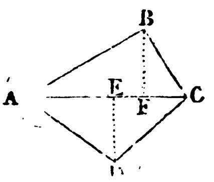
PROBLEM

PROBLEM IV.

To Measure a Four-sided Field.

1. *By the Chain.*

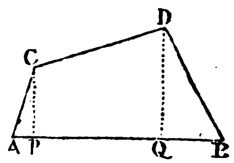
AE	214	210	DE
AF	362	306	BF
AC	592		



Measure along one of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last problem; or else the sides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

Otherwise, by the Chain.

AP	110	352	PC
AQ	745	595	QD
AB	1110		



Measure, on the longest side, the distances AP, AQ, AB; and the perpendiculars PC, QD.

2. *By taking some of the Angles.*

Measure the diagonal AC (see the last fig. but one), and the angles CAB, CAD, ACB, ACD.—Or measure the four sides, and any one of the angles, as BAD.

Thus.		Or thus.	
AC	591	AB	486
CAB	37° 20′	BC	394
CAD	41 15	CD	410
ACB	72 25	DA	462
ACD	54 40	BAD	78° 35′

PROBLEM V.

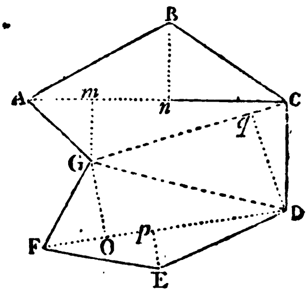
To Survey any Field by the Chain only.

HAVING set up marks at the corners, where necessary, of the proposed field ABCDEFG, walk over the ground, and consider how it can best be divided in triangles and trapeziums; and measure them separately, as in the last two problems. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the first trapezium, beginning at A, measure the diagonal AC, and the two

two perpendiculars Gm, Bn. Then the base GC, and the perpendicular Dq. Lastly, the diagonal DF, and the two perpendiculars pE, oG. All which measures write against the corresponding parts of a rough figure drawn to resemble the figure surveyed, or set them down in any other form you choose.

Thus.

Am	135	130	mG
An	410	180	nB
AC	550		
cq	152	230	qD
CG	440		
FO	237	120	OG
Fp	288	80	pE
FD	520		



Or thus.

Measure all the sides AB, BC, CD, DE, EF, FG, GA; and the diagonals AC, CG, GD, DF.

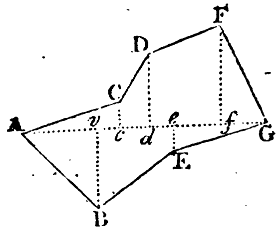
Otherwise.

Many pieces of land may be very well surveyed, by measuring any base line, either within or without them, with the perpendiculars let fall on it from every corner. For they are by those means divided into several triangles and trepezoids, all whose parallel sides are perpendicular to the base line; and the sum of these triangles and trapeziums will be equal to the figure proposed if the base line fall within it; if not, the sum of the parts which are without being taken from the sum of the whole which are both within and without, will leave the area of the figure proposed.

In pieces that are not very large, it will be sufficiently exact to find the points, in the base line, where the several perpendiculars will fall, by means of the *cross*, or even by judging by the eye only, and from thence measuring to the corners for the lengths of the perpendiculars.—And it will be most convenient to draw the line so as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and measuring along the line AG, the distances and perpendiculars on the right and left are as below.

Ab	315	350	bB
AC	440	70	cC
Ad	585	320	dD
Ae	610	50	eE
Af	990	470	fF
AG	1020	0	

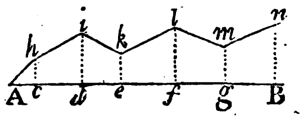


PROBLEM VI.

To Measure the Offsets.

Ahiklmn being a crooked hedge, or brook, &c. From A measure in a straight direction along the side of it to B. And in measuring along this line AB, observe when you are directly opposite any bends or corners of the boundary, as at c, d, e, &c; and from these measure the perpendicular offsets ch, di, &c, with the offset-staff, if they are not very large, otherwise with the chain itself; and the work is done. The register, or field-book, may be as follows:

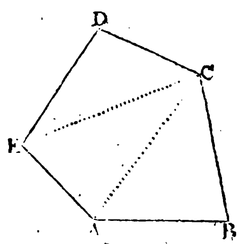
Offs.left.		Base line AB	
	0	⊙	A
ch	62	45	Ac
di	84	220	Ad
ek	70	340	Ae
fl	98	510	Af
gm	57	634	Ag
Bn	91	785	AB



PROBLEM VII.

To Survey any Field with the Plain Table.

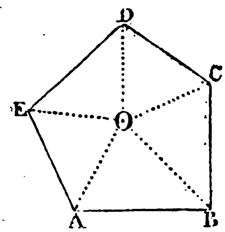
1. *From one Station.*



PLANT the table at any angle as C, from which all the other angles, or marks set up, can be seen; turn the table about till the needle point to the flower-de-luce; and there screw it fast. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the sights you see the mark D: and by the edge of the index draw a dry or obscure line: then measure the distance CD, and lay that distance down on the line CD. Then turn the index about the point C, till the mark E be seen through the sights,

sights, by which draw a line, and measure the distance to E, laying it on the line from C to E. In like manner determine the positions of CA and CB, by turning the sights successively to A and B; and lay the lengths of those lines down. Then connect the points, by drawing the black lines CD, DE, EA, AB, BC, for the boundaries of the field.

2. *From a Station Within the Field.*



When all the other parts cannot be seen from one angle, choose some place O within, or even without, if more convenient, from which the other parts can be seen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index successively to O, turning it round with the sights to each angle, A, B, C, D, E, drawing dry lines to them by the edge of the index; then measuring the distances OA, OB, &c, and laying them down on those lines. Lastly, draw the boundaries AB, BC, CD, DE, EA.

3. *By going Round the Figure.*

When the figure is a wood, or water, or when from some other obstruction you cannot measure lines across it; begin at any point A, and measure around it, either within or without the figure, and draw the directions of all the sides, thus: Plant the table at A; turn it with the needle to the north or flower-de-luce; fix it, and mark the point A. Apply the index to A, turning it till you can see the point E, and there draw a line: then the point B, and there draw a line: then measure these lines, and lay them down from A to E and B. Next move the table to B, lay the index along the line AB, and turn the table about till you can see the mark A, and screw fast the table; in which position also the needle will again point to the flower-de-luce, as it will do indeed at every station when the table is in the right position. Here turn the index about B till through the sights you see the mark C; there draw a line, measure BC, and lay the distance on that line after you have set down the table at C. Turn it then again into its proper position, and in like manner find the next line CD. And so on quite around by E, to A again. Then the proof of the work will be the joining at A: for if the work be all right, the last direction EA on the ground, will pass exactly through the point A on the paper; and the measured distance will also reach exactly to A. If these do not coincide, or nearly so, some error has been committed, and the work must be examined over again.

PROBLEM VIII.

To Survey a Field with the Theodolite, &c.

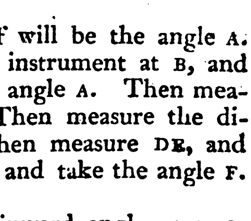
1. *From One Point or Station.*

WHEN all the angles can be seen from one point, as the angle C (first fig. to last prob.) place the instrument at C, and turn it about, till through the fixed sights you see the mark B, and there fix it. Then turn the moveable index about till the mark A be seen through the sights, and note the degrees cut on the instrument. Next turn the index successively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD. Lastly measure the lines CB, CA, CE, CD; and enter the measures in a field-book, or rather against the corresponding parts of a rough figure drawn by guess to resemble the field.

2. *From a Point Within or Without.*

Plant the instrument at O (last. fig.), and turn it about till the fixed sights point to any object, as A; and there screw it fast. Then turn the moveable index round till the sights point successively to the other points E, D, C, B, noting the degrees cut off at each of them; which gives all the angles round the point O. Lastly measure the distances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

3. *By going Round the Field.*



By measuring round, either within or without the field, proceed thus. Having set up marks at B, C, &c, near the corners as usual, plant the instrument at any point A, and turn it till the fixed index be in the direction AB, and there screw it fast: then turn the moveable index to the direction AF; and the degrees cut off will be the angle A. Measure the line AB, and plant the instrument at B, and there in the same manner observe the angle A. Then measure BC, and observe the angle C. Then measure the distance CD, and take the angle D. Then measure DE, and take the angle E. Then measure EF, and take the angle F. And lastly measure the distance FA.

To prove the work; add all the inward angles A, B, C, &c, together; for when the work is right, their sum will be equal to twice as many right angles as the figure has sides, wanting 4 right angles. But when there is an angle, as F, that bends inwards, and you measure the external angle, which

which is less than two right angles, subtract it from 4 right angles, or 360 degrees, to give the internal angle greater than a semicircle or 180 degrees.

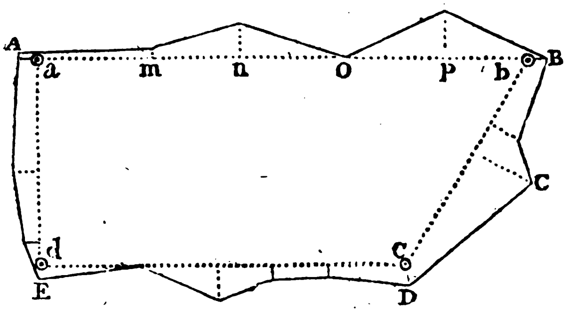
Otherwise.

Instead of observing the internal angles, we may take the external angles, formed without the figure by producing the sides farther out. And in this case, when the work is right, their sum altogether will be equal to 360 degrees. But when one of them, as F, runs inwards, subtract it from the sum of the rest, to leave 360 degrees.

PROBLEM IX.

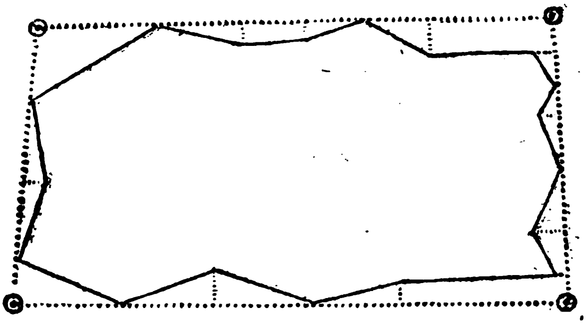
To Survey a Field with Crooked Hedges, &c.

WITH any of the instruments, measure the lengths and positions of imaginary lines running as near the sides of the field as you can; and, in going along them, measure the offsets in the manner before taught; then you will have the plan on the paper in using the plain table, drawing the crooked hedges through the ends of the offsets; but in surveying with the theodolite, or other instrument, set down the measures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.



So, in surveying the piece ABCDE, set up marks, a, b, c, d, dividing it so as to have as few sides as may be. Then begin at any station, a, and measure the lines ab, bc, cd, da, taking their positions, or the angles, a, b, c, d; and, in going along the lines, measure all the offsets, as at m, n, o, p, &c, along every station-line.

And this is done either within the field, or without, as may be most convenient. When there are obstructions within, as wood, water, hills, &c, then measure without, as in the next following figure.



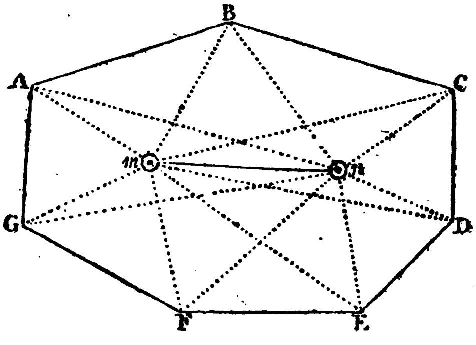
PROBLEM X.

To Survey a Field, or any other Thing, by Two Stations.

THIS is performed by choosing two stations from which all the marks and objects can be seen; then measuring the distance between the stations, and at each station taking the angles formed by every object from the station line or distance.

The two stations may be taken either within the bounds, or in one of the sides, or in the direction of two of the objects, or quite at a distance and without the bounds of the objects or part to be surveyed.

In this manner, not only grounds may be surveyed, without even entering them, but a map may be taken of the principal parts of a county, or the chief places of a town, or any part of a river or coast surveyed, or any other inaccessible objects; by taking two stations, on two towers, or two hills, or such-like.



PROBLEM XI.

To Survey a Large Estate.

IF the estate be very large, and contain a great number of fields, it cannot well be done by surveying all the fields singly,

singly, and then putting them together; nor can it be done by taking all the angles and boundaries that enclose it. For in these cases, any small errors will be so much increased, as to render it very much distorted. But proceed as below.

1. Walk over the estate two or three times, in order to get a perfect idea of it, or till you can keep the figure of it pretty well in mind. And to help your memory, draw an eye-draught of it on paper, or at least of the principal parts of it, to guide you; setting the names within the fields in that draught.

2. Choose two or more eminent places in the estate, for stations, from which all the principal parts of it can be seen: selecting these stations as far distant from one another as convenient.

3. Take such angles, between the stations, as you think necessary, and measure the distances from station to station, always in a right line: these things must be done, till you get as many angles and lines as are sufficient for determining all the points of station. And in measuring any of these station-distances, mark accurately where these lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c; and where any remarkable object is placed, by measuring its distance from the station-line; and where a perpendicular from it cuts that line. And thus as you go along any main station-line, take offsets to the ends of all hedges, and to any pond, house, mill, bridge, &c, noting every thing down that is remarkable.

4. As to the inner parts of the estate, they must be determined, in like manner, by new station-lines: for, after the main stations are determined, and every thing adjoining to them, then the estate must be subdivided into two or three parts by new station-lines; taking inner stations at proper places, where you can have the best view. Measure these station-lines as you did the first, and all their intersections with hedges, and offsets to such objects as appear. Then proceed to survey the adjoining fields, by taking the angles that the sides make with the station-line, at the intersections, and measuring the distances to each corner, from the intersections. For the station-lines will be the bases to all the future operations; the situation of all parts being entirely dependent on them; and therefore they should be taken of as great length as possible; and it is best for them to run along some of the hedges or boundaries of one or more fields, or to pass through some of their angles. All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide till at last you *come to single fields*: repeating the same work for the inner

stations as for the outer ones, till all is done; and close the work as often as you can, and in as few lines as possible.

5. An estate may be so situated that the whole cannot be surveyed together; because one part of the estate cannot be seen from another. In this case, you may divide it into three or four parts, and survey the parts separately, as if they were lands belonging to different persons; and at last join them together.

6. As it is necessary to protract or lay down the work as you proceed in it, you must have a scale of a due length to do it by. To get such a scale, measure the whole length of the estate in chains; then consider how many inches long the map is to be; and from these will be known how many chains you must have in an inch; then make the scale accordingly, or choose one already made.

PROBLEM XII.

To Survey a County, or large Tract of Land.

1. CHOOSE two, three, or four eminent places, for stations; such as the tops of high hills or mountains, towers, or church steeples, which may be seen from one another; from which most of the towns and other places of note may also be seen; and so as to be as far distant from one another as possible. On these places raise beacons, or long poles, with flags of different colours flying at them, so as to be visible from all the other stations.

2. At all the places which you would set down in the map, plant long poles, with flags at them of several colours, to distinguish the places from one another; fixing them on the tops of church steeples, or the tops of houses; or in the centres of smaller towns and villages.

These marks then being set up at a convenient number of places, and such as may be seen from both stations; go to one of these stations, and, with an instrument to take angles, standing at that station, take all the angles between the other station and each of these marks. Then go to the other station, and take all the angles between the first station and each of the former marks, setting them down with the others, each against its fellow with the same colour. You may, if convenient, also take the angles at some third station, which may serve to prove the work, if the three lines intersect in that point where any mark stands. The marks must stand till the observations are finished at both stations; and then they may be taken down, and set up at new places. The same operations must be performed at both stations, for *these new places*; and the like for others. The instrument for

for taking angles must be an exceeding good one, made on purpose with telescopic sights, and of a good length of radius.

3. And though it be not absolutely necessary to measure any distance, because, a stationary line being laid down from any scale, all the other lines will be proportional to it; yet it is better to measure some of the lines, to ascertain the distances of places in miles, and to know how many geometrical miles there are in any length; as also from thence to make a scale to measure any distance in miles. In measuring any distance, it will not be exact enough to go along the high roads; which, by reason of their turnings and windings, hardly ever lie in a right line between the stations; which must cause endless reductions, and require great trouble to make it a right line; for which reason it can never be exact. But a better way is to measure in a straight line with a chain, between station and station, over hills and dales, or level fields and all obstacles. Only in case of water, woods, towns, rocks, banks, &c, where we cannot pass, such parts of the line must be measured by the methods of inaccessible distances; and besides, allowing for ascents and descents, when they are met with. A good compass, that shows the bearing of the two stations, will always direct us to go straight, when the two stations cannot be seen; and in the progress, if we can go straight, offsets may be taken to any remarkable places, likewise noting the intersection of the station-line with all roads, rivers, &c.

4. From all the stations, and in the whole progress, we must be very particular in observing sea-coasts, river-mouths, towns, castles, houses, churches, mills, trees, rocks, sands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, sluices, floodgates, locks, &c, and in general every thing that is remarkable.

5. After we have done with the first and main station-lines, which command the whole county; we must then take inner stations, at some places already determined; which will divide the whole into several partitions: and from these stations we must determine the places of as many of the remaining towns as we can. And if any remain in that part, we must take more stations, at some places already determined; from which we may determine the rest. And thus go through all the parts of the county, taking station after station, till we have determined the whole. And in general the station-distances must always pass through such remarkable points as have been determined before, by the former stations.

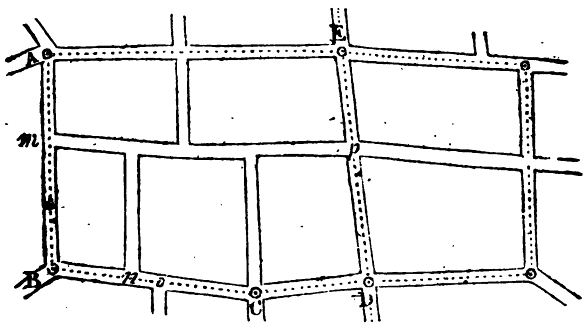
PROBLEM

PROBLEM XIII.

To Survey a Town or City.

THIS may be done with any of the instruments for taking angles, but best of all with the plain table, where every minute part is drawn while in sight. Instead of the common surveying or Gunter's chain, it will be best, for this purpose, to have a chain 50 feet long, divided into 50 links of one foot each, and an offset-staff of 10 feet long.

Begin at the meeting of two or more of the principal streets, through which we can have the longest prospects, to get the longest station-lines: there having fixed the instrument, draw lines of direction along those streets, using two men as marks, or poles set in wooden pedestals, or perhaps some remarkable places in the houses at the farther ends, as windows, doors, corners, &c. Measure these lines with the chain, taking offsets with the staff, at all corners of streets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houses, &c. Then remove the instrument to another station, along one of these lines; and there repeat the same process as before. And so on till the whole is finished.



Thus, fix the instrument at A, and draw lines in the direction of all the streets meeting there; then measure AB, noting the street on the left at m. At the second station B, draw the directions of the streets meeting there; and measure from B to C, noting the places of the streets at n and o as you pass by them. At the third station C, take the direction of all the streets meeting there, and measure CD. At D do the same, and measure DE, noting the place of the cross streets at P. And in this manner go through all the principal streets. This done, proceed to the smaller and intermediate streets; and lastly to the lanes, alleys, courts, yards, 'd every part that it may be thought proper to represent in plan

PROBLEM

PROBLEM XIV.

To lay down the Plan of any Survey

If the survey was taken with the plain table, we have a rough plan of it already on the paper which covered the table. But if the survey was with any other instrument, a plan of it is to be drawn from the measures that were taken in the survey; and first of all a rough plan on paper.

To do this, you must have a set of proper instruments, for laying down both lines and angles, &c; as scales of various sizes, the more of them, and the more accurate, the better, scales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal scales are best for the lines, because they extend to three figures, or chains, and links, which are 100 parts of chains. But in using the diagonal scale, a pair of compasses must be employed, to take off the lengths of the principal lines very accurately. But a scale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of those offsets on the station-line; which is done at only one application of the edge of the scale to that line, and then pricking off all at once the distances along it. Angles are to be laid down, either with a good scale of chords, which is perhaps the most accurate way, or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all lines and angles must be laid down on the plan in the same order in which they were measured in the field, and in which they are written in the field-book; laying down first the angles for the position of lines, next the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themselves, all with dry or obscure lines; then a black line drawn through the extremities of all the offsets, will be the edge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or close properly, proceed next to the smaller objects, till you have entered every thing that ought to appear in the plan, as houses, brooks, trees, hills, gates, stiles, roads, lanes, mills, bridges, woodlands, &c, &c.

The north side of a map or plan is commonly placed uppermost, and a meridian is somewhere drawn, with the compass or flower-de-luce pointing north. Also, in a vacant part, a scale of equal parts or chains is drawn, with the title of the map in conspicuous characters, and embellished with a compartment. Hills are shadowed, to distinguish them in the map. *Colour the* hedges with different colours; repre-

sent hilly grounds by broken hills and valleys; draw single dotted lines for foot-paths, and double ones for horse or carriage roads. Write the name of each field and remarkable place within it, and, if you choose, its contents in acres, roods, and perches.

In a very large estate, or a county, draw vertical and horizontal lines through the map, denoting the spaces between them by letters placed at the top, and bottom, and sides, for readily finding any field or other object mentioned in a table.

In mapping counties, and estates that have uneven grounds of hills and valleys, reduce all oblique lines, measured uphill and down-hill, to horizontal straight lines, if that was not done during the survey, before they were entered in the field-book, by making a proper allowance to shorten them. For which purpose there is commonly a small table engraven on some of the instruments for surveying.

THE NEW METHOD OF SURVEYING.

PROBLEM XV.

To Survey and Plan by the New Method.

IN the former method of measuring a large estate, the accuracy of it depends both on the correctness of the instruments, and on the care in taking the angles. To avoid the errors incident to such a multitude of angles, other methods have of late years been used by some few skilful surveyors: the most practical, expeditious, and correct, seems to be the following, which is performed, without taking angles, by measuring with the chain only.

Choose two or more eminences, as grand stations, and measure a principal base line from one station to another; noting every hedge, brook, or other remarkable object, as you pass by it; measuring also such short perpendicular lines to the bends of hedges as may be near at hand. From the extremities of this base line, or from any convenient parts of the same, go off with other lines to some remarkable object situated towards the sides of the estate, without regarding the angles they make with the base line or with one another; still remembering to note every hedge, brook, or other object, that you pass by. These lines, when laid down by intersections, will, with the base line, form a grand triangle on the estate; several of which, if need be, being thus measured and laid down, you may proceed to form other smaller triangles and trapezoids on the sides of the former: and so on till you finish with the enclosures individually. By which means a kind of skeleton of the estate may first be obtained, and

	580	to v
40	500	
76	300	
F 76	100	
	395	to F
J 20	150	
	854	J
15	850	
	730	to E
30	400	
	340	60
0	260	
I 20	170	50
	744	to H
	672	0
70	450	0
a 50	25	
	1260	to s
32	1000	
	880	
	780	32
	590	40
	570	I
	530	40
	376	H
	256	150 leave off
	200	64
G	144	130
	2676	G
	2676	30
	896	24
	632	
	620	50
180 from u towards s	588	F
	644	to f
D	488	32
	2260	
	2250	E
20	2240	
36	[illegible]	
	[illegible]	
	[illegible]	20 u w
	[illegible]	160
	[illegible]	[illegible]
	[illegible]	[illegible]
	660	

	768	to A
	526	70
	496	
70	460	
40	124	
D	100	
	455	D
	400	76
C	48	10
	600	to r
	432	C
56	160	
B 44	36	
B	152	to q
	480	B
p 24	160	
o	1750	
	1600	44 to s
	1028	
	930	A
44	666	
70	310	z
d 60	236	
	2248	480 to b
	1950	y
	1836	
28	1724	
60	1600	
30	1480	x
0	1320	
50	1110	
	1080	
	840	w
u	750	50
	4440	36
	4420	v
	3884	u
	3380	60
	2992	90
	2692	t
120	2624	
	2592	
	2500	s leave off
	2070	56
	1900	
	1840	r
60	1770	
	1320	q
	808	p
leave off 40	650	
80	360	
o 20	170	
h produce by i	220	o
	190	46

		580	to v
	.10	500	
	78	300	
F	76	100	

		395	to F
J	20	190	

		954	J
	15	850	

		730	to E
	30	490	
		340	60
	0	280	
I	20	170	50

		744	to H
		672	0
	70	450	0
a	50	25	

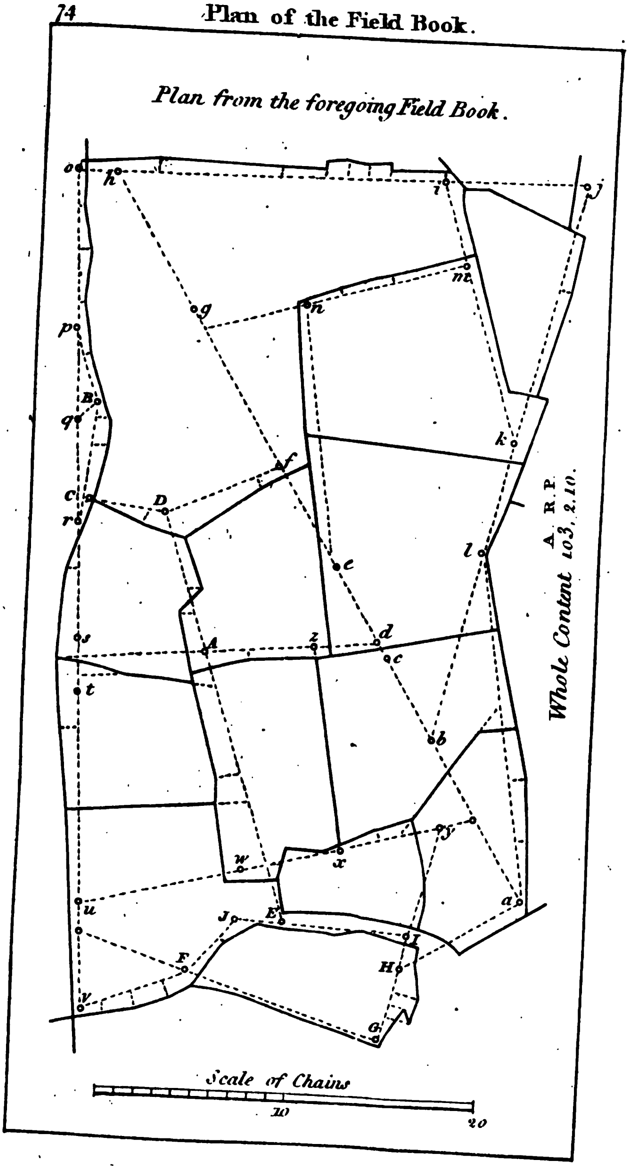
		1160	to v
	32	1000	
		890	
		780	32
		590	40
		570	I
		530	40
		376	H
		256	150 leave off
		190	64
G		144	130

		1676	G
		1676	30
		896	24
		632	
		620	50
180 from u towards v		588	F

		644	to f
D		488	32

		2260	
		2250	E
	20	2210	
	56	2050	
		2030	

		1970	230 to w
		1552	280
		1380	96
		950	110
		860	54


Plan from the foregoing Field Book.
Whole Content A. R. P. 103. 2. 10.
Scale of Chains
10
20

and the chief lines serve as the bases of such triangles and trapezoids as are necessary to fill up all the interior parts.

The field-book is ruled into three columns, as usual. In the middle one are set down the distances on the chain-line, at which any mark, offset, or other observation, is made; and in the right and left hand columns are entered the offsets and observations made on the right and left hand respectively of the chain-line; sketching on the sides the shape or resemblance of the fences or boundaries.

It is of great advantage, both for brevity and perspicuity, to begin at the bottom of the leaf, and write upwards; denoting the crossing of fences, by lines drawn across the middle column, or only a part of such a line on the right and left opposite the figures, to avoid confusion; and the corners of fields, and other remarkable turns in the fences where offsets are taken to, by lines joining in the manner the fences do; as will be best seen by comparing the book with the plan annexed to the field-book following, p. 74.

The letter in the left-hand corner at the beginning of every line, is the mark or place measured *from*; and that at the right-hand corner at the end, is the mark measured *to*: But when it is not convenient to go exactly from a mark, the place measured from is described *such a distance* from *one mark* towards *another*; and where a former mark is not measured to, the exact place is ascertained by saying, turn to the right or left hand, *such a distance to such a mark*, it being always understood that those distances are taken in the chain-line.

The characters used are, ⌈ for *turn to the right hand*, ⌉ for *turn to the left hand*, and ⌃ placed over an offset, to show that it is not taken at right angles with the chain-line, but in the direction of some straight fence; being chiefly used when crossing their directions; which is a better way of obtaining their true places than by offsets at right angles.

When a line is measured whose position is determined, either by former work (as in the case of producing a given line, or measuring from one known place or mark to another) or by itself (as in the third side of the triangle), it is called a *fast line*, and a double line across the book is drawn at the conclusion of it; but if its position is not determined (as in the second side of the triangle), it is called a *loose line*, and a single line is drawn across the book. When a line becomes determined in position, and is afterwards continued farther, a double line half through the book is drawn.

When a loose line is measured, it becomes absolutely necessary to measure some other line that will determine its position. Thus, the first line *ab* or *bb*, being the base of

triangl

triangle, is always determined; but the position of the second side *hj*, does not become determined, till the third side *jb* is measured; then the position of both is determined, and the triangle may be constructed.

At the beginning of a line, to fix a loose line to the mark or place measured from, the sign of turning to the right or left hand must be added, as at *h* in the second, and *j* in the third line; otherwise a stranger, when laying down the work, may as easily construct the triangle *hjb* on the wrong side of the line *ab*, as on the right one: but this error cannot be fallen into, if the sign above named be carefully observed.

In choosing a line to fix a loose one, care must be taken that it does not make a very acute or obtuse angle; as in the triangle *p*B*r*, by the angle at B being very obtuse, a small deviation from truth, even the breadth of a point at *p* or *r*, would make the error at B, when constructed, very considerable; but by constructing the triangle *p*B*q*, such a deviation is of no consequence.

Where the words *leave off* are written in the field-book, it signifies that the taking of offsets is from thence discontinued; and of course something is wanting between that and the next offset, to be afterwards determined by measuring some other line.

The field-book for this method, and the plan drawn from it, are contained in the four following pages, engraven on copper plates; answerable to which the pupil is to draw a plan from the measures in the field-book, of a larger size, viz. to a scale of a double size will be convenient, such a scale being also found on most instruments. In doing this, begin at the commencement of the field-book, or bottom of the first page, and draw the first line *ab* in any direction at pleasure, and then the next two sides of the first triangle *bhj* by sweeping intersected arcs; and so all the triangles in the same manner, after each other in their order; and afterwards setting the perpendicular and other offsets at their proper places, and through the ends of them drawing the bounding fences.

Note. That the field-book begins at the bottom of the first page, and reads up to the top; hence it goes to the bottom of the next page, and to the top; and thence it passes from the bottom of the third page to the top, which is the end of the field-book. The several marks measured to or from, are here denoted by the letters of the alphabet, first the small ones, *a*, *b*, *c*, *d*, &c, and after them the capitals *A*, *B*, *C*, *D*, &c. But, instead of these letters, some surveyors use the numbers in order, 1, 2, 3, 4, &c.

OF

OF THE OLD KIND OF FIELD-BOOK.

In surveying with the plain table, a field-book is not used, as every thing is drawn on the table immediately when it is measured. But in surveying with the theodolite, or any other instrument, some kind of a field-book must be used, to write down in it a register or account of all that is done and occurs relative to the survey in hand.

This book every one contrives and rules as he thinks fittest for himself. The following is a specimen of a form which has been formerly used. It is ruled in three columns, as below.

Here $\odot$ 1 is the first station, where the angle or bearing is 105° 25′. On the left, at 73 links in the distance or principal line, is an offset of 92; and at 610 an offset of 24 to a cross hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the first line, the 0 denotes its terminating in the hedge. And so on for the other stations.

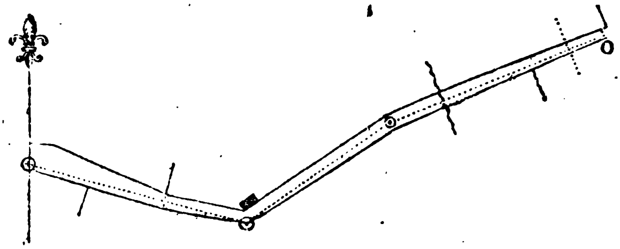
A line is drawn under the work, at the end of every station line, to prevent confusion.

Form of this Field-Book.

Offsets and Remarks on the left.	Stations, Bearings, and Distances.	Offsets and Remarks on the right.
	$\odot$ 1	
	105° 25′	
80	00	25 corner
92	73	
	248	Brown's hedge
a cross hedge 24	610	35
	954	00
	$\odot$ 2	
	53° 10′	
house corner 51	25	21
	120	29 a tree
34	734	40 a stile
	$\odot$ 3	
	67° 20′	
	61	35
a brook 30	248	
	639	16 a spring
foot path 16	810	
cross hedge 18	973	20 a pond

Th

Then the plan, on a small scale drawn from the above field-book, will be as in the following figure. But the pupil may draw a plan of 3 or 4 times the size on his paper book. The dotted lines denote the 3 chain or measured lines, and the black lines the boundaries on the right and left.

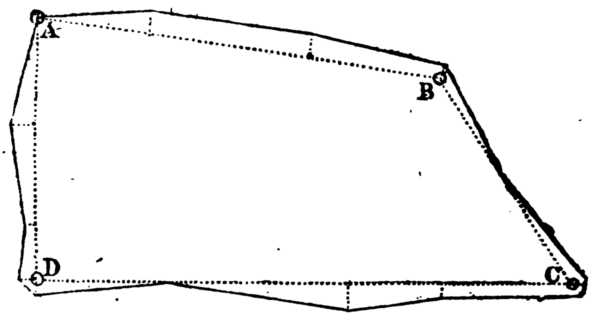


But some skilful surveyors now make use of a different method for the field-book, namely, beginning at the bottom of the page and writing upwards; sketching also a neat boundary on either hand, resembling the parts near the measured lines as they pass along; an example of which was given in the new method of surveying, in the preceding pages.

In smaller surveys and measurements, a good way of setting down the work, is to draw by the eye, on a piece of paper, a figure resembling that which is to be measured; and so writing the dimensions, as they are found, against the corresponding parts of the figure. And this method may be practised to a considerable extent, even in the larger surveys.

Another specimen of a field-book, with its plan, is as follows; being a single field, surveyed with the chain, and the theodolite for taking angles; which the pupil will likewise draw of a larger size.

	⊙ A			⊙ C	
	82° 55′			57° 10′	
0	0		35	268	
40	230		50	479	
48	572		0	846	
30	860		30	1140	
	⊙ B			⊙ D	
	130° 35′		40	0	
0	238		25	117	
20	520		45	312	
			0	554	



SECTION III.

OF COMPUTING AND DIVIDING.

PROBLEM XVI.

To Compute the Contents of Fields.

1. COMPUTE the contents of the figures as divided into triangles, or trapeziums, by the proper rules for these figures laid down in measuring; multiplying the perpendiculars by the diagonals or bases, both in links, and divide by 2; the quotient is acres, after having cut off five figures on the right for decimals. Then bring these decimals to roods and perches, by multiplying first by 4, and then by 40. An example of which is given in the description of the chain, pag. 53.

2. In small and separate pieces, it is usual to compute their contents from the measures of the lines taken in surveying them, without making a correct plan of them.

3. In pieces bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most accurately measured separately as small trapezoids.

4. Sometimes such pieces as that last mentioned, are computed by finding a mean breadth, by adding all the offsets together, and dividing the sum by the number of them, accounting that for one of them where the boundary meets the station-line, (which increases the number of them by 1, for the divisor, though it does not increase the sum or quantity to be divided); then multiply the length by that mean breadth.

5. But in larger pieces and whole estates, consisting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents, quite independent of the measures of the lines and angles that were taken in surveying. For then new lines are drawn in the fields

fields on the plans, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale from which it was drawn, and so multiplied together for the contents. In this way, the work is very expeditiously done, and sufficiently correct; for such dimensions are taken as afford the most easy method of calculation; and among a number of parts, thus taken and applied to a scale, though it be likely that some of the parts will be taken a small matter too little, and others too great, yet they will, on the whole, in all probability, very nearly balance one another, and give a sufficiently accurate result. After all the fields and particular parts are thus computed separately, and added all together into one sum; calculate the whole estate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these also together. Then if this sum be equal to the former, or nearly so, the work is right; but if the sums have any considerable difference, it is wrong, and they must be examined, and re-computed, till they nearly agree.

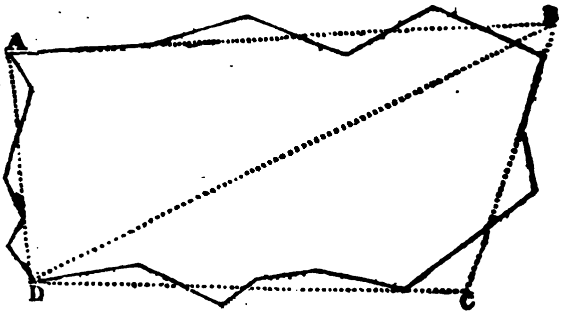
6. But the chief art in computing, consists in finding the contents of pieces bounded by curved or very irregular lines, or in reducing such crooked sides of fields or boundaries to straight lines, that shall inclose the same or equal area with those crooked sides, and so obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked sides to straight ones, is very easily and accurately performed in this manner:—Apply the straight edge of a thin, clear piece of lanthorn-horn to the crooked line, which is to be reduced, in such a manner, that the small parts cut off from the crooked figure by it, may be equal to those which are taken in: which equality of the parts included and excluded you will presently be able to judge of very nicely by a little practice: then with a pencil, or point of a tracer, draw a line by the straight edge of the horn. Do the same by the other sides of the field or figure. So shall you have a straight-sided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the crooked figure proposed.

Or, instead of the straight edge of the horn, a horse-hair, or fine thread, may be applied across the crooked sides in the same manner; and the easiest way of using the thread, is to string a small slender bow with it, either of wire, or cane, or whale-bone, or such-like slender elastic matter; for, the bow keeping it always stretched, it can be easily and neatly applied with one hand, while the other is at liberty to make two marks by the side of it, to draw the straight line by.

EXAMPLE.

EXAMPLE.

Thus, let it be required to find the contents of the same figure as in Prob. IX, page 65, to a scale of 4 chains to an inch.



Draw the 4 dotted straight lines AB, BC, CD, DA, cutting off equal quantities on both sides of them, which they do as near as the eye can judge: so is the crooked figure reduced to an equivalent right-lined one of 4 sides, ABCD. Then draw the diagonal BD, which, by applying a proper scale to it, measures suppose 1256. Also the perpendicular, or nearest distance from A to this diagonal, measures 456; and the distance of C from it, is 428.

Then, half the sum of 456 and 428, multiplied by the diagonal 1256, gives 555152 square links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

As a general example of this practice, let the contents be computed of all the fields separately in the foregoing plan facing page 75, and, by adding the contents altogether, the whole sum or content of the estate will be found nearly equal to $103\frac{1}{4}$ acres. Then, to prove the work, divide the whole plan into two parts, by a pencil line drawn across it any way near the middle, as from the corner *l* on the right, to the corner near *s* on the left; then, by computing these two large parts separately, their sum must be nearly equal to the former sum, when the work is all right.

PROBLEM XVII.

To Transfer a Plan to Another Paper, &c.

AFTER the rough plan is completed, and a fair one is wanted; this may be done by any of the following methods.

First

First Method.—Lay the rough plan on the clean paper, keeping them always pressed flat and close together, by weights laid on them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them asunder, and connect the pricked points on the clean paper, with lines; and it is done. This method is only to be practised in plans of such figures as are small and tolerably regular, or bounded by right lines.

Second Method.—Rub the back of the rough plan over with black-lead powder; and lay this blacked part on the clean paper on which the plan is to be copied, and in the proper position. Then, with the blunt point of some hard substance, as brass, or such-like, trace over the lines of the whole plan; pressing the tracer so much, as that the black lead under the lines may be transferred to the clean paper: after which, take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink—Or, instead of blacking the rough plan, we may keep constantly a blacked paper to lay between the plans.

Third Method.—Another method of copying plans, is by means of squares. This is performed by dividing both ends and sides of the plan which is to be copied into any convenient number of equal parts, and connecting the corresponding points of division with lines: which will divide the plan into a number of small squares. Then divide the paper, on which the plan is to be copied, into the same number of squares, each equal to the former when the plan is to be copied of the same size, but greater or less than the others, in the proportion in which the plan is to be increased or diminished, when of a different size. Lastly, copy into the clean squares the parts contained in the corresponding squares of the old plan; and you will have the copy, either of the same size, or greater or less in any proportion.

Fourth Method.—A fourth method is by the instrument called a pentagraph, which also copies the plan in any size required.

Fifth Method.—But the neatest method of any, at least in copying from a fair plan, is this. Procure a copying frame or glass, made in this manner; namely, a large square of the best window glass, set in a broad frame of wood, which can be raised up to any angle, when the lower side of it rests on a table. Set this frame up to any angle before you, facing a strong light; fix the old plan and clean paper together, with al pins quite around, to keep them together, the clean

paper

paper being laid uppermost, and over the face of the plan to be copied. Lay them, with the back of the old plan, on the glass; namely, that part which you intend to begin at to copy first; and by means of the light shining through the papers, you will very distinctly perceive every line of the plan through the clean paper. In this state then trace all the lines on the paper with a pencil. Having drawn that part which covers the glass, slide another part over the glass, and copy it in the same manner. Then another part. And so on, till the whole is copied. Then take them asunder, and trace all the pencil lines over with a fine pen and Indian ink, or with common ink. And thus you may copy the finest plan, without injuring it in the least.

OF ARTIFICERS' WORKS,

AND

TIMBER MEASURING.

1. OF THE CARPENTER'S OR SLIDING RULE.

THE Carpenter's or Sliding Rule, is an instrument much used in measuring of timber and artificers' works, both for taking the dimensions, and computing the contents.

The instrument consists of two equal pieces, each a foot in length, which are connected together by a folding joint.

One side or face of the rule is divided into inches, and eighths, or half-quarters. On the same face also are several plane scales, divided into twelfth parts by diagonal lines; which are used in planning dimensions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into ten equal parts, and each of these into ten parts again: so that by means of this last scale, dimensions are taken in feet, tenths, and hundredths, and multiplied as common decimal numbers, which is the best way.

On the one part of the other face are four lines, marked A, B, C, D; the two middle ones B and C being on a slider, which runs in a groove made in the stock. The same numbers serve for both these two middle lines, the one being above the numbers, and the other below.

 These

These four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or lowest line, D, is a single one, proceeding from 4 to 40. It is also called the girt line, from its use in computing the contents of trees and timber; and on it are marked WG at 17·15, and AG at 18·95, the wine and ale gage points, to make this instrument serve the pupose of a gaging rule.

On the other part of this face, there is a table of the value of a load, or 50 cubic feet of timber, at all prices, from 6 pence to 2 shillings a foot.

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; but when 1 at the beginning is counted 10, then the 1 in the middle is 100, and the 10 at the end 1000; and so on. And all the smaller divisions are altered proportionally.

II. ARTIFICERS' WORK.

ARTIFICERS compute the contents of their works by several different measures. As,

Glazing and masonry, by the foot; Painting, plastering, paving, &c, by the yard, of 9 square feet: Flooring, partitioning, roofing, tiling, &c, by the square of 100 square feet:

And brickwork, either by the yard of 9 square feet, or by the perch, or square rod or pole, containing $272\frac{1}{4}$ square feet, or $30\frac{1}{4}$ square yards, being the square of the rod or pole of $16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards long.

As this number $272\frac{1}{4}$ is troublesome to divide by, the $\frac{1}{4}$ is often omitted in practice, and the content in feet divided only by the 272.

All works, whether superficial or solid, are computed by the rules proper to the figure of them, whether it be a triangle, or rectangle, a parallelopiped, or any other figure.

III. BRICKLAYERS' WORK.

BRICKWORK is estimated at the rate of a brick and a half thick. So that if a wall be more or less than this standard thickness, it must be reduced to it, as follows:

Multiply the superficial content of the wall by the number of half bricks in the thickness, and divide the product by 3.

The

The dimensions of a building may be taken by measuring half round on the outside and half round it on the inside; the sum of these two gives the compass of the wall, to be multiplied by the height, for the content of the materials.

Chimneys are commonly measured as if they were solid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them. All windows, doors, &c, are to be deducted out of the contents of the walls in which they are placed.

EXAMPLES.

EXAM. 1. How many yards and rods of standard brick-work are in a wall whose length or compass is 57 feet 3 inches, and height 24 feet 6 inches; the wall being $2\frac{1}{2}$ bricks or 5 half bricks thick? Ans. 8 rods, $17\frac{1}{3}$ yards.

EXAM. 2. Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick?

Ans. 169·753 yards.

EXAM. 3. A triangular gable is raised $17\frac{1}{2}$ feet high, on an end wall whose length is 24 feet 9 inches, the thickness being 2 bricks: required the reduced content?

Ans. $32·08\frac{1}{3}$ yards.

EXAM. 4. The end wall of a house is 28 feet 10 inches long, and 55 feet 8 inches high, to the eaves; 20 feet high is $2\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is $1\frac{1}{2}$ brick thick; above which is a triangular gable, of 1 brick thick, which rises 42 courses of bricks, of which every 4 courses make a foot. What is the whole content in standard measure? Ans. 253·626 yards.

IV. MASONS' WORK.

To Masonry belong all sorts of stone-work; and the measure made use of is a foot, either superficial or solid.

Walls, columns, blocks of stone or marble, &c, are measured by the cubic foot; and pavements, slabs, chimney-pieces, &c, by the superficial or square foot.

Cubic or solid measure is used for the materials, and square measure for the workmanship.

In the solid measure, the true length, breadth, and thickness are taken and multiplied continually together. In the superficial, there must be taken the length and breadth of every part of the projection which is seen without the general upright face of the building.

EXAMPLES.

EXAM. 1. Required the solid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick?

Ans. 1310¾ feet.

EXAM. 2. What is the solid content of a wall, the length being 24 feet 3 inches, height 10 feet 9 inches, and 2 feet thick? Ans. 521·375 feet.

EXAM. 3. Required the value of a marble slab, at 8s. per foot; the length being 5 feet 7 inches, and breadth 1 foot 10 inches? Ans. 4l. 1s. 10½d.

EXAM. 4. In a chimney piece, suppose the

length of the mantle and slab, each	4 feet	6 inches
breadth of both together -	3	2
length of each jamb - -	4	4
breadth of both together -	1	9

Required the superficial content? Ans. 21 feet 10 inches.

V. CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood-work of a house, such as flooring, partitioning, roofing, &c.

Large and plain articles are usually measured by the square foot or yard, &c; but enriched mouldings, and some other articles, are often estimated by running or lineal measure; and some things are rated by the piece.

In measuring of Joists, take the dimensions of one joist, and multiply its content by the number of them; considering that each end is let into the wall about ⅔ of the thickness, as it ought to be.

Partitions are measured from wall to wall for one dimension, and from floor to floor, as far as they extend, for the other.

The measure of Centering for Cellars is found by making a string pass over the surface of the arch for the breadth, and taking the length of the cellar for the length: but in groin centering, it is usual to allow double measure, on account of their extraordinary trouble.

In Roofing, the dimensions, as to length, breadth, and depth, are taken as in flooring joists, and the contents computed the same way.

In Floor-boarding, take the length of the room for one dimension, and the breadth for the other, to multiply together for the content.

For Stair-cases, take the breadth of all the steps, by making a line

a line ply close over them, from the top to the bottom, and multiply the length of this line by the length of a step, for the whole area.—By the length of a step is meant the length of the front and the returns at the two ends; and by the breadth is to be understood the girts of its two outer surfaces, or the tread and riser.

For the Balustrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-post, for the one dimension; and twice the length of the baluster on the landing, with the girt of the hand-rail, for the other dimension.

For Wainscoting, take the compass of the room for the one dimension; and the height from the floor to the ceiling, making the string ply close into all the mouldings, for the other.

For Doors, take the height and the breadth, to multiply them together for the area.—If the door be panneled on both sides, take double its measure for the workmanship; but if one side only be panneled, take the area and its half for the workmanship.—*For the Surrounding Architrave*, girt it about the uttermost part for its length; and measure over it, as far as it can be seen when the door is open, for the breadth.

Window-shutters, *Bases*, &c, are measured in like manner.

In measuring of Joiners' work, the string is made to ply close into all mouldings, and to every part of the work over which it passes.

EXAMPLES.

EXAM. 1. Required the content of a floor, 48 feet 6 inches long, and 24 feet 3 inches broad? Ans. 11 sq. $76\frac{1}{8}$ feet.

EXAM. 2. A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many squares are in it?
Ans. 5 sq. $98\frac{1}{8}$ feet.

EXAM. 3. How many squares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning? Ans. 18·3973 squares.

EXAM. 4. What cost the roofing of a house at 10*s*. 6*d*. a square; the length within the walls being 52 feet 8 inches, and the breadth 30 feet 6 inches; reckoning the roof $\frac{3}{2}$ of the flat? Ans. 12*l*. 12*s*. $11\frac{1}{4}$*d*.

EXAM.

EXAM. 5. To how much, at 6*s*. per square yard, amounts the wainscoting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compass 83 feet 8 inches; also the three window-shutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the doors and shutters, being worked on both sides, are reckoned work and half work?

Ans. 36*l*. 12*s*. $2\frac{1}{4}$*d*.

VI. SLATERS' AND TILERS' WORK.

IN these articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of slates at the bottom, or for how much one row of slates or tiles is laid over another.

When the roof is of a true pitch, that is, forming a right angle at top; then the breadth of the building, with its half added, is the girt over both sides nearly.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney shafts or window holes.

EXAMPLES.

EXAM. 1. Required the content of a slated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches?

Ans. $174\frac{5}{18}$ yards.

EXAM. 2. To how much amounts the tiling of a house, at 25*s*. 6*d*. per square; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches; also the eaves projecting 16 inches on each side, and the roof of a true pitch?

Ans. 24*l*. 9*s*. $5\frac{3}{4}$*d*.

VII. PLASTERERS' WORK.

PLASTERERS' work is of two kinds; namely, ceiling, which is plastering on laths; and rendering, which is plastering on walls: which are measured separately.

The

The contents are estimated either by the foot or the yard, or the square, of 100 feet. Inriched mouldings, &c, are rated by running or lineal measure.

Deductions are made for chimneys, doors, windows, &c.

EXAMPLES.

EXAM. 1. How many yards contains the ceiling which is 43 feet 3 inches long, and 25 feet 6 inches broad?

Ans. 122½.

EXAM. 2. To how much amounts the ceiling of a room, at 10*d.* per yard; the length being 21 feet 8 inches, and the breadth 14 feet 10 inches? - Ans. 1*l.* 9*s.* 8¾*d.*

EXAM. 3. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8*d.* and the latter at 3*d.* per yard; allowing for the door of 7 feet by 3 feet 8, and a fire-place of 5 feet square?

Ans. 1*l.* 13*s.* 3¼*d.*

EXAM. 4. Required the quantity of plastering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under side of the cornice, which girts 8½ inches, and projects 5 inches from the wall on the upper part next the ceiling; deducting only for a door 7 feet by 4?

Ans.	53 yards	5 feet	3½ inches	of rendering
	18	5	6	of ceiling
		39	$0\frac{1}{12}$	of cornice.

VIII. PAINTERS' WORK.

PAINTERS' work is computed in square yards. Every part is measured where the colour lies; and the measuring line is forced into all the mouldings and corners.

Windows are done at so much a piece. And it is usual to allow double measure for carved mouldings, &c.

EXAMPLES.

EXAM. 1. How many yards of painting contains the room which is 65 feet 6 inches in compass, and 12 feet 4 inches high? Ans. $89\frac{41}{54}$ yards.

EXAM. 2. The length of a room being 20 feet, its breadth 14 feet

14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Ans. $73\frac{2}{27}$ yards.

Exam. 3. What cost the painting of a room, at 6*d*. per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; also the door is 7 feet by 3 feet 6, and the window-shutters to two windows each 7 feet 9 by 3 feet 6; but the breaks of the windows themselves are 8 feet 6 inches high, and 1 foot 3 inches deep; including also the window cills or seats, and the soffits above, the dimensions of which are known from the other dimensions: but deducting the fire-place of 5 feet by 5 feet 6? Ans. 3*l*. 3*s*. $10\frac{3}{4}$*d*.

IX. GLAZIERS' WORK.

Glaziers take their dimensions, either in feet, inches, and parts, or feet, tenths, and hundredths. And they compute their work in square feet.

In taking the length and breadth of a window, the cross bars between the squares are included. Also windows of round or oval forms are measured as square, measuring them to their greatest length and breadth, on account of the waste in cutting the glass.

EXAMPLES.

Exam. 1. How many square feet contains the window which is 4·25 feet long, and 2·75 feet broad? Ans. $11\frac{2}{3}$.

Exam. 2. What will the glazing a triangular sky-light come to, at 10*d*. per foot; the base being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches? Ans. 1*l*. 15*s*. $1\frac{3}{4}$*d*.

Exam. 3. There is a house with three tiers of windows, three windows in each tier, their common breadth 3 feet 11 inches:

now the height of the first tier is	7 feet	10 inches
of the second	6	8
of the third	5	4

Required the expence of glazing at 14*d*. per foot? Ans. 13*l*. 11*s*. $10\frac{1}{2}$*d*.

Exam.

EXAM. 4. Required the expense of glazing the windows of a house at 13*d.* a foot; there being three stories, and three windows in each story:

the height of the lower tier is	7 feet	9 inches
of the middle	6	6
of the upper	5	$3\frac{1}{4}$
and of an oval window over the door	1	$10\frac{1}{2}$

the common breadth of all the windows being 3 feet 9 inches? Ans. 12*l.* 5*s.* 6*d.*

X. PAVERS' WORK.

PAVERS' work is done by the square yard. And the content is found by multiplying the length by the breadth.

EXAMPLES.

EXAM. 1. What cost the paving a foot path, at 3*s.* 4*d.* a yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches? Ans. 5*l.* 7*s.* $11\frac{1}{2}$*d.*

EXAM. 2. What cost the paving a court, at 3*s.* 2*d.* per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? Ans. 7*l.* 4*s.* $5\frac{1}{4}$*d.*

EXAM. 3. What will be the expense of paving a rectangular court-yard, whose length is 63 feet, and breadth 45 feet; in which there is laid a foot-path of 5 feet 3 inches broad, running the whole length, with broad stones, at 3*s.* a yard; the rest being paved with pebbles at 2*s.* 6*d.* a yard?

Ans. 40*l.* 5*s.* $10\frac{1}{2}$*d.*

XI. PLUMBERS' WORK.

PLUMBERS' work is rated at so much a pound, or else by the hundred weight of 112 pounds.

Sheet lead, used in roofing, guttering, &c, is from 6 to 10lb. to the square foot. And a pipe of an inch bore is commonly 13 or 14 lb. to the yard in length.

EXAMPLES.

EXAM. 1. How much weighs the lead which is 39 feet

6 inches

6 inches long, and 3 feet 3 inches broad, at $8\frac{1}{2}$lb. to the square foot? Ans. $1091\frac{3}{16}$lb.

Exam. 2. What cost the covering and guttering a roof with lead, at 18*s*. the cwt; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former 9·831 lb. and the latter 7·373 lb. to the square foot? Ans. 115*l*. 9*s*. $1\frac{1}{2}$*d*.

XII. TIMBER MEASURING.

PROBLEM I.

To find the Area, or Superficial Content, of a Board or Plank.

Multiply the length by the mean breadth.

Note. When the board is tapering, add the breadths at the two ends together, and take half the sum for the mean breadth. Or else take the mean breadth in the middle.

By the Sliding Rule.

Set 12 on B to the breadth in inches on A; then against the length in feet on B, is the content on A, in feet and fractional parts.

EXAMPLES.

Exam. 1. What is the value of a plank, at $1\frac{1}{2}$*d*. per foot, whose length is 12 feet 6 inches, and mean breadth 11 inches? Ans. 1*s*. 5*d*.

Exam. 2. Required the content of a board, whose length is 11 feet 2 inches, and breadth 1 foot 10 inches?
Ans. 20 feet 5 inches 8″.

Exam. 3. What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at $2\frac{1}{2}$*d*. a foot.
Ans. 3*s*. $3\frac{3}{4}$*d*.

Exam. 4. Required the value of 5 oaken planks at 3*d*. per foot, each of them being $17\frac{1}{2}$ feet long; and their several breadths as follows, namely, two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower? Ans. 1*l*. 5*s*. $9\frac{1}{4}$*d*.

PROBLEM

PROBLEM II.

To find the Solid Content of Squared or Four-sided Timber.

MULTIPLY the mean breadth by the mean thickness, and the product again by the length, for the content nearly.

By the Sliding Rule.

C D D C
As length : 12 or 10 : : quarter girt : solidity.

That is, as the length in feet on C, is to 12 on D, when the quarter girt is in inches, or to 10 on D, when it is in tenths of feet; so is the quarter girt on D, to the content on C.

Note 1. If the tree taper regularly from the one end to the other; either take the mean breadth and thickness in the middle, or take the dimensions at the two ends, and half their sum will be the mean dimensions: which multiplied as above, will give the content nearly.

2. If the piece do not taper regularly, but be unequally thick in some parts and small in others; take several different dimensions, add them all together, and divide their sum by the number of them, for the mean dimensions.

EXAMPLES.

EXAM. 1. The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and less end 1 foot 6 inches and 1 foot 3 inches, and the thickness at the greater and less end 1 foot 3 inches and 1 foot; required the solid content? Ans. 28 feet 7 inches.

EXAM. 2 What is the content of the piece of timber, whose length is $24\frac{1}{2}$ feet, and the mean breadth and thickness each 1·04 feet? Ans. $26\frac{1}{2}$ feet.

EXAM. 3. Required the content of a piece of timber, whose length is 20·38 feet, and its ends unequal squares, the sides of the greater being $19\frac{1}{8}$ inches, and the side of the less $9\frac{7}{8}$ inches? Ans. 29·7562 feet.

EXAM.

Exam. 4. Required the content of the piece of timber, whose length is 27·36 feet; at the greater end the breadth is 1·78, and thickness 1·23; and at the less end the breadth is 1·04, and thickness 0·91 feet? Ans. 41·278 feet.

PROBLEM III.

To find the Solidity of Round or Unsquared Timber.

Multiply the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference, by the length, for the content.

By the Sliding Rule.

As the length upon C : 12 or 10 upon D ::
quarter girt, in 12ths or 10ths, on D : content on C.

Note 1. When the tree is tapering, take the mean dimensions as in the former problems, either by girting it in the middle, for the mean girt, or at the two ends, and taking half the sum of the two; or by girting it in several places, then adding all the girts together, and dividing the sum by the number of them, for the mean girt. But when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

2. This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or nearly what the quantity would be, after the tree is hewed square in the usual way: so that it seems intended to make an allowance for the squaring of the tree.

EXAMPLES.

Exam. 1. A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content? Ans. 116$\frac{1}{3}$ feet.

Exam. 2. The length of a tree is 24 feet, its girt at the thicker end 14 feet, and at the smaller end 2 feet; required the content? Ans. 96 feet.

Exam. 3. What is the content of a tree, whose mean girt is 3·15 feet, and length 14 feet 6 inches? Ans. 8·9922 feet.

Exam. 4. Required the content of a tree; whose length is 17$\frac{1}{4}$ feet, which girts in five different places as follows, namely, in the first place 9·43 feet, in the second 7·92, in the third 6·15, in the fourth 4·74, and in the fifth 3·16? Ans. 42·519525.

CONIC

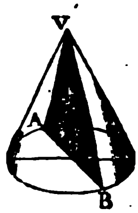
CONIC SECTIONS.

DEFINITIONS.

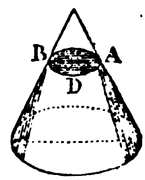
1. CONIC SECTIONS are the figures made by a plane cutting a cone.

2. According to the different positions of the cutting plane, there arise five different figures or sections, namely, a triangle, a circle, an ellipsis, an hyperbola, and a parabola: the three last of which only are peculiarly called Conic Sections.

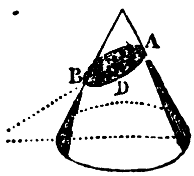
3. If the cutting plane pass through the vertex of the cone, and any part of the base, the section will evidently be a triangle; as VAB.



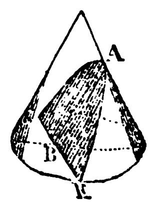
4. If the plane cut the cone parallel to the base, or make no angle with it, the section will be a circle; as ABD.



5. The section DAB is an ellipse when the cone is cut obliquely through both sides, or when the plane is inclined to the base in a less angle than the side of the cone is.

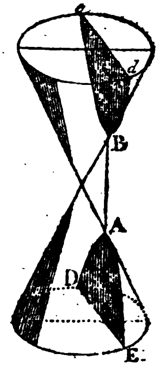


6. The section is a parabola, when the cone is cut by a plane parallel to the side, or when the cutting plane and the side of the cone make equal angles with the base.



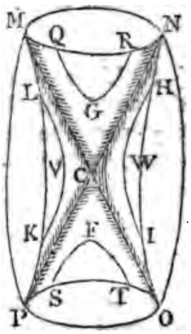
7. The

7. The section is an hyperbola, when the cutting plane makes a greater angle with the base than the side of the cone makes.



8. And if all the sides of the cone be continued through the vertex, forming an opposite equal cone, and the plane be also continued to cut the opposite cone, this latter section will be the opposite hyperbola to the former; as dBe.

And further, if there be four cones CMN, COP, CMP, CNO, having all the same vertex C, and all their axes in the same plane, and their sides touching or coinciding in the common intersecting lines MCO, NCP; then if these four cones be all cut by one plane, parallel to the common plane of their axes, there will be formed the four hyperbolas GQR, FST, VKL, WHI, of which each two opposites are equal, and the other two are conjugates to them; as here in the annexed figure, and the same as represented in the two following pages.



9. The Vertices of any section, are the points where the cutting plane meets the opposite sides of the cone, or the sides of the vertical triangular section; as A and B.

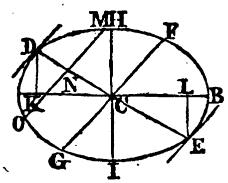
Hence the ellipse and the opposite hyperbolas, have each two vertices; but the parabola only one; unless we consider the other as at an infinite distance.

10. The Axis, or Transverse Diameter, of a conic section, is the line or distance AB between the vertices.

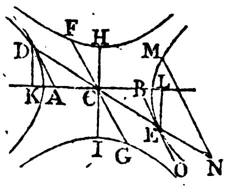
Hence the axis of a parabola is infinite in length, Ab being only a part of it.

Ellipse.

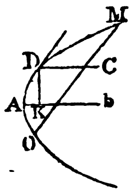
Ellipse.



Hyperbolas.



Parabola.



11. The Centre C is the middle of the axis.

Hence the centre of a parabola is infinitely distant from the vertex. And of an ellipse, the axis and centre lie within the curve; but of an hyperbola, without.

12. A Diameter is any right line, as AB or DE, drawn through the centre, and terminated on each side by the curve; and the extremities of the diameter, or its intersections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. And hence also every diameter of the ellipse and hyperbola have two vertices; but of the parabola, only one; unless we consider the other as at an infinite distance.

13. The Conjugate to any diameter, is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So, FG, parallel to the tangent at D, is the conjugate to DE; and HI, parallel to the tangent at A, is the conjugate to AB.

Hence the conjugate HI, of the axis AB, is perpendicular to it.

14. An Ordinate to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So DK, EL, are ordinates to the axis AB; and MN, NO, ordinates to the diameter DE.

Hence the ordinates of the axis are perpendicular to it.

15. An Absciss is a part of any diameter contained between its vertex and an ordinate to it; as AK or BK, or DN or EN.

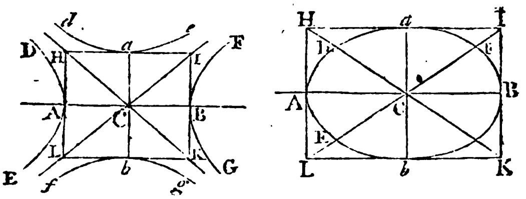
Hence, in the ellipse and hyperbola, every ordinate has two determinate abscisses; but in the parabola, only one; the other vertex of the diameter being infinitely distant.

16. The Parameter of any diameter, is a third proportional to that diameter and its conjugate.

17. The

17. The Focus is the point in the axis where the ordinate is equal to half the parameter. As K and L, where DK or EL is equal to the semi-parameter. The name focus being given to this point from the peculiar property of it mentioned in the corol. to theor. 9 in the Ellipse and Hyperbola following, and to theor. 6 in the Parabola.

Hence, the ellipse and hyperbola have each two foci; but the parabola only one.



18. If DAE, FBG, be two opposite hyperbolas, having AB for their first or transverse axis, and ab for their second or conjugate axis. And if dae, fbg, be two other opposite hyperbolas having the same axes, but in the contrary order, namely, ab their first axis, and AB their second; then these two latter curves dae, fbg, are called the conjugate hyperbolas to the two former DAE, FBG; and each pair of opposite curves mutually conjugate to the other; being all cut by one plane, from four conjugate cones, as in page 94, def. 8.

19. And if tangents be drawn to the four vertices of the curves, or extremities of the axes, forming the inscribed rectangle HIKL; the diagonals HCK, ICL, of this rectangle, are called the asymptotes of the curves. And if these asymptotes intersect at right angles, or the inscribed rectangle be a square, or the two axes AB and ab be equal, then the hyperbolas are said to be right-angled, or equilateral.

SCHOLIUM.

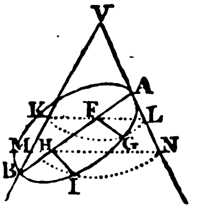
The rectangle inscribed between the four conjugate hyperbolas, is similar to a rectangle circumscribed about an ellipse, by drawing tangents, in like manner, to the four extremities of the two axes; and the asymptotes or diagonals in the hyperbola, are analogous to those in the ellipse, cutting this curve in similar points, and making that pair of conjugate diameters which are equal to each other. Also, the whole figure formed by the four hyperbolas, is, as it were, an ellipse turned inside out, cut open at the extremities D, E, F, G, of the said equal conjugate diameters, and those four points drawn out to an infinite distance; the curvature being turned the contrary way, but the axes, and the rectangle passing through their extremities, continuing fixed.

Of

OF THE ELLIPSE.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.



LET AVB be a plane passing through the axis of the cone; AGIH another section of the cone perpendicular to the plane of the former; AB the axis of this elliptic section; and FG, HI, ordinates perpendicular to it. Then it will be, as $FG^2 : HI^2 :: AF . FB : AH . HB$.

For, through the ordinates FG, HI, draw the circular sections KGL, MIN, parallel to the base of the cone, having KL, MN, for their diameters, to which FG, HI, are ordinates, as well as to the axis of the ellipse.

Now, by the similar triangles AFL, AHN, and BFK, BHM,

it is $AF : AH :: FL : HN$,
and $FB : HB :: KF : MH$;

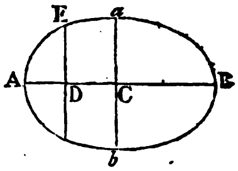
hence, taking the rectangles of the corresponding terms, it is, the rect. $AF . FB : AH . HB :: KF . FL : MH . HN$.

But, by the circle, $KF . FL = FG^2$, and $MH . HN = HI^2$;
Therefore the rect. $AF . FB : AH . HB :: FG^2 : HI^2$. Q. E. D.

THEOREM II.

As the Square of the Transverse Axis :
Is to the Square of the Conjugate ::
So is the Rectangle of the Abscisses :
To the Square of their Ordinate.

That is, $AB^2 : ab^2$ or
$AC^2 : ac^2 :: AD \cdot DB : DE^2$.



For, by theor. 1, $AC \cdot CB : AD \cdot DB :: ca^2 : DE^2$;
But, if C be the centre, then $AC \cdot CB = AC^2$, and Ca is the semi-conjugate.
Therefore $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q. E. D.

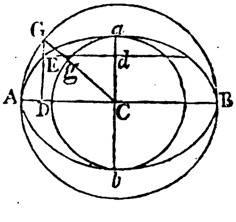
Corol. Or, by div. $AB : \frac{ab^2}{AB} :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$,
that is, $AB : p :: AD \cdot DB$ or $CA^2 - CD^2 : DE^2$;
where p is the parameter $\frac{ab^2}{AB}$, by the definition of it.

That is, As the transverse,
Is to its parameter,
So is the rectangle of the abscisses,
To the square of their ordinate.

THEOREM III.

As the Square of the Conjugate Axis:
Is to the Square of the Transverse Axis::
So is the Rectangle of the Abscisses of the Conjugate, or the Difference of the Squares of the Semi-conjugate and Distance of the Centre from any Ordinate of that Axis:
To the Square of their Ordinate.

That is,
$ca^2 : CB^2 :: ad \cdot db$ or $ca^2 - cd^2 : dE^2$.



For, draw the ordinate ED to the transverse AB.
Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD \cdot DB$ or $CA^2 - CD^2$,
or - - - - - $ca^2 : CA^2 :: cd^2 : CA^2 - dE^2$.
But - - - - $ca^2 : CA^2 :: ca^2 : CA^2$,
theref. by subtr. $ca^2 : CA^2 :: ca^2 - cd^2$ or $ad \cdot db : dE^2$.

Q. E. D.

Corol.

Corol. 1. If two circles be described on the two axes as diameters, the one inscribed within the ellipse, and the other circumscribed about it; then an ordinate in the circle will be to the corresponding ordinate in the ellipse, as the axis of this ordinate, is to the other axis.

That is, $CA : ca :: DG : DE$,
and $ca : CA :: dg : dE$.

For, by the nature of the circle, $AD \cdot DB = DG^2$; theref. by the nature of the ellipse, $CA^2 : ca^2 :: AD \cdot DB$ or $DG^2 : DE^2$,
or $CA : ca :: DG : DE$.
In like manner - $ca : CA :: dg : dE$.
Also, by equality, - $DG : DE$ or $cd :: dE$ or $DC : dg$.
Therefore cgG is a continued straight line.

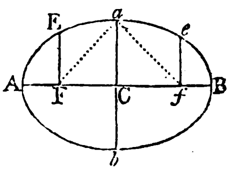
Corol. 2. Hence also, as the ellipse and circle are made up of the same number of corresponding ordinates, which are all in the same proportion of the two axes, it follows that the areas of the whole circle and ellipse, as also of any like parts of them, are in the same proportion of the two axes, or as the square of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipse, are as the square of each axis and the rectangle of the two; and therefore the ellipse is a mean proportional between the two circles.

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Difference of the Squares of the Semi-axes;

Or, the Square of the Distance between the Foci, is equal to the Difference of the Squares of the two Axes.

That is, $CF^2 = CA^2 - ca^2$,
or $Ff^2 = AB^2 - ab^2$.



For, to the focus F draw the ordinate FE; which, by the definition, will be the semi-parameter. Then, by the nature of the curve - - $CA^2 : ca^2 :: CA^2 - CF^2 : FE^2$;
and by the def. of the para. $CA^2 : ca^2 :: ca^2 : FE^2$;
therefore - - $ca^2 = CA^2 - CF^2$;
and by addit. and subtr. $CF^2 = CA^2 - ca^2$;
or, by doubling, - $Ff^2 = AB^2 - ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle CFa; and the distance Fa from the focus to the extremity of the conjugate axis, is = AC the semi-transverse.

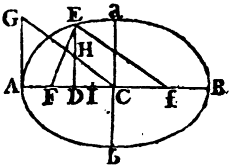
Corol. 2. The conjugate semi-axis Ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For $Ca^2 = CA^2 - CF^2 = CA + CF \cdot CA - CF = AF \cdot FB$.

THEOREM V.

The Sum of two Lines drawn from the two Foci to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,
$FE + fe = AB$.



For, draw AG parallel and equal to Ca the semi-conjugate; and join CG meeting the ordinate DE in H; also take CI a 4th proportional to CA, CF, CD.

Then, by theor. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$;
and, by sim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
consequently $DE^2 = AG^2 - DH^2 = Ca^2 - DH^2$.

Also $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
And, by right-angled triangles, $FE^2 = FD^2 + DE^2$;
therefore $FE^2 = CF^2 + Ca^2 - 2CF \cdot CD + CD^2 - DH^2$.

But by theor. 4, $CF^2 + Ca^2 = CA^2$,
and by supposition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 - DH^2$.

Again, by supp. $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore - $CI^2 = CD^2 - DH^2$;
consequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.

And the root or side of this square is $FE = CA - CI = AI$.

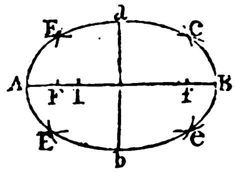
In the same manner it is found that $fE = CA + CI = BI$.
Conseq. by addit. $FE + fE = AI + BI = AB$. - Q. E. D.

Corol.

Corol. 1. Hence CI or $eA - FE$ is a 4th proportional to CA, CF, CD.

Corol. 2. And $fE - FE = 2CI$; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD.

Corol. 3. Hence is derived the common method of describing this curve mechanically by points, or with a thread, thus:



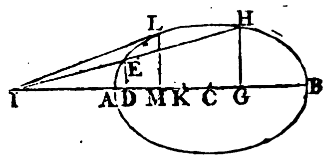
In the transverse take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, describe arcs intersecting in E, which will be a point in the curve. In like manner, assuming other points I, as many other points will be found in the curve. Then with a steady hand, the curve line may be drawn through all the points of intersection E.

Or, take a thread of the length AB of the transverse axis, and fix its two ends in the foci F, f, by two pins. Then carry a pen or pencil round by the thread, keeping it always stretched, and its point will trace out the curve line.

THEOREM VI.

If from any Point I in the Axis produced, a Line IL be drawn touching the Curve in one Point L; and the Ordinate LM be drawn; and if C be the Centre or Middle of AB: Then shall CM be to CI as the Square of AM to the Square of AI.

That is,
$CM : CI :: AM^2 : AI^2$.



For, from the point I draw any other line IEH to cut the curve in two points E and H; from which let fall the perpendiculars ED and HG; and bisect DG in K.

Then, by theo. 1, $AD \cdot DB : AG \cdot GB : DE^2 : GH^2$,
and by sim. triangles, $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 \cdot IG^2$.

But $DB = CB + CD = AC + CD = AG + DC - CG = 2CK + AG$,
and $GB = CB - CG = AC - CG = AD + DC - CG = 2CK + AD$;
theref. $AD \cdot 2CK + AD \cdot AG : AG \cdot 2CK + AD \cdot AG :: ID^2 : IG^2$,
and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$,

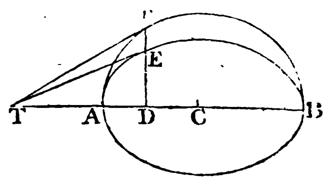
or - $2CK : 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$,
or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$;
theref. by div. $CK : IK :: AD \cdot AG : ID^2 - AD \cdot \overline{2IK}$,
and, by comp. $CK : IC :: AD \cdot AG : ID^2 - AD \cdot \overline{ID + IA}$,
or - $CK : CI :: AD \cdot AG : AI^2$.

But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes $CM : CI :: AM^2 : AI^2$. Q. E. D.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Centre.

That is,
CA is a mean proportional between CD and CT;
or CD, CA, CT, are continued proportionals.



For, by theor. 6, $CD : CT :: AD^2 : AT^2$.
that is, $CD : CT :: (CA - CD)^2 : (CT - CA)^2$,
or - $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$,
and - $CD : DT :: CD^2 + CA^2 : CT^2 - CD^2$,
or - $CD : DT :: CD^2 + CA^2 : (CT + CD)\, DT$,
or - $CD^2 : CD \cdot CT :: CD^2 + CA^2 : CD \cdot DT + CT \cdot DT$,
hence - $CD^2 : CA^2 :: CD \cdot DT : CT \cdot DT$,
and - $CD^2 : CA^2 :: CD : CT$.
therefore (th. 78, Geom.) $CD : CA :: CA : CT$. Q. E. D.

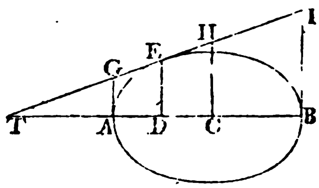
Corol. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every ellipse described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from these four Points, namely, the Centre, the two Extremities of the Axis, and the Point of Contact; those Four Perpendiculars will be Proportionals.

That is,
AG : DE : : CH : BI.



For, by theor. 7,	TC : AC : : AC : DC,
theref. by div.	TA : AD : : TC : AC or CB,
and by comp.	TA : TD : : TC : TB,
and by sim. tri.	AG : DE : : CH : BI. Q. E. D.

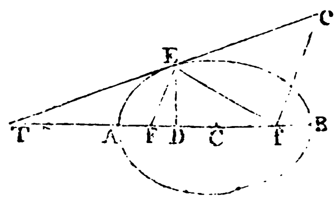
Corol. Hence $\left.\begin{array}{ll} & TA, TD, TC, TB \\ \text{and} & TG, TE, TH, TI \end{array}\right\}$ are also proportionals.

For these are as AG, DE, CH, BI, by similar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two Lines will make equal Angles with the Tangent.

That is,
the $\angle$FET = $\angle$fEe.



For, draw the ordinate DE, and fe parallel to FE.

By cor. 1, theor. 5,	CA : CD : : GF : CA − FE,
and by theor. 7,	CA : CD : : CT : CA;
therefore	CT : CF : : CA : CA − FE;
and by add. and sub.	TF : Tf : : FE : 2CA − FE or fE by th. 5.
But by sim. tri.	TF : Tf : : Fe : fe;
therefore	fE = fe, and conseq. $\angle$e = $\angle$fEe.
But, because	FE is parallel to fe, the $\angle$e = $\angle$FET;
therefore the	$\angle$FET = $\angle$fEe. Q. E. D.

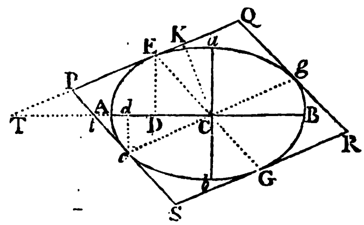
Corol.

Corol. As opticians find that the angle of incidence is equal to the angle of reflexion, it appears from this theorem, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from those points to the other focus. So the ray fE is reflected into FE. And this is the reason why the points F, f, are called the *foci*, or burning points.

THEOREM X.

All the Parallelograms circumscribed about an Ellipse are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS =
the rectangle AB . ab



Let EG, eg, be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four less and equal parallelograms. Also, draw the ordinates DE, de, and CK perpendicular to PQ; and let the axis CA produced meet the sides of the parallelogram, produced if necessary, in T and t.

Then, by theor 7,	CT : CA :: CA : CD,
and	ct : CA :: CA : cd;
theref. by equality,	CT : ct :: cd : CD;
but, by sim. triangles,	CT : ct :: TD : cd,
theref. by equality,	TD : cd :: cd : CD,
and the rectangle	TD . DC is = the square cd^2.
Again, by theor. 7,	CD : CA :: CA : CT,
or, by division,	CD : CA :: DA : AT,
and by composition,	CD : DB :: AD : DT;
conseq. the rectangle	CD . DT = cd^2 = AD . DB*.
But, by theor. 1,	$CA^2 : Ca^2 ::$ (AD . DB or) $cd^2 : DE^2$,
therefore	CA : Ca :: cd : DE;

* *Corol.* Because $cd^2 = AD . DB = CA^2 - CD^2$,
therefore $CA^2 = CD^2 + cd^2$.
In like manner, $ca^2 = DE^2 + de^2$.

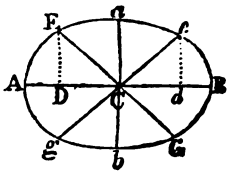
In

In like manner, $CA : ca :: CD : de$,
or $ca : de :: CA : CD$.
But, by theor. 7, $CT : CA :: CA : CD$;
theref. by equality, $CT : CA :: ca : de$.
But, by sim. tri. $CT : CK :: ce : de$;
theref. by equality, $CK : CA :: ca : ce$,
and the rectangle $CK \cdot ce = CA \cdot ca$.
But the rect. $CK \cdot ce =$ the parallelogram $CEPe$,
theref. the rect. $CA \cdot ca =$ the parallelogram $CEPe$,
conseq. the rect. $AB \cdot ab =$ the parallelogram $PQRS$. Q. E. D.

THEOREM XI.

The Sum of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely, the Sum of the Squares of the two Axes.

That is,
$AB^2 + ab^2 = EG^2 + eg^2$;
where EG, eg, are any pair of conjugate diameters.



For, draw the ordinates ED, ed.
Then, by cor. to theor. 10, $CA^2 = CD^2 + cd^2$,
and - - - $ca^2 = DE^2 + de^2$;
therefore the sum $CA^2 + ca^2 = CD^2 + DE^2 + cd^2 + de^2$.
But, by right-angled Δs, $CE^2 = CD^2 + DE^2$,
and - - - $ce^2 = cd^2 + de^2$;
therefore the sum $CE^2 + ce^2 = CD^2 + DE^2 + cd^2 + de^2$.
consequently - - $CA^2 + ca^2 = CE^2 + ce^2$;
or, by doubling, $AB^2 + ab^2 = EG^2 + eg^2$. Q. E. D.

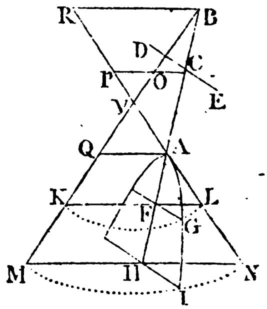
Note. All these theorems in the Ellipse, and their demonstrations, are the very same, word for word, as the corresponding number of those in the Hyperbola, next following, having only sometimes the word *sum* changed for the word *difference*.

OF

Of the HYPERBOLA.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.



Let AVB be a plane passing through the vertex and axis of the opposite cones; AGIH another section of them perpendicular to the plane of the former; AB the axis of the hyperbolic sections; and FG, HI, ordinates perpendicular to it. Then it will be, as $FG^2 : HI^2 :: AF . FB : AH . HB$.

For, through the ordinates FG, HI, draw the circular sections KGL, MIN, parallel to the base of the cone, having KL, MN, for their diameters, to which FG, HI, are ordinates, as well as to the axis of the hyperbola.

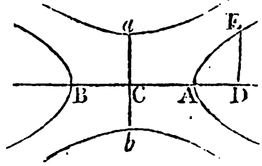
Now, by the similar triangles AFL, AHN, and BFK, BHM,
it is $AF : AH :: FL : HN$,
and $FB : HB :: KF : MH$;
hence, taking the rectangles of the corresponding terms,
it is, the rect. $AF . FB : AH . HB :: KF . FL : MH . HN$.

But, by the circle, $KF . FL = FG^2$, and $MH . HN = HI^2$;
Therefore the rect. $AF . FB : AH . HB :: FG^2 : HI^2$.

Q. E. D.

THEOREM II.

As the Square of the Transverse Axis :
Is to the Square of the Conjugate : :
So is the Rectangle of the Abscisses :
To the Square of their Ordinate.



That is, $AB^2 : ab^2$ or
$AC^2 :: ac^2 :: AD . DB : DE^2$.

For, by theor. 1, $AC . CB : AD . DB :: Ca^2 : DE^2$;
But, if C be the centre, then $AC . CB = AC^2$, and ca is the semi-conj.

Therefore - $AC^2 : AD . DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD . DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD . DB : DE^2$. Q. E. D.

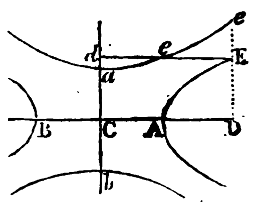
Corol. Or, by div. $AB : \frac{ab^2}{AB} :: AD . DB$ or $CD^2 - CA^2 : DE^2$,
that is, $AB : p :: AD . DB$ or $CD^2 - CA^2 : DE^2$;
where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the transverse,
Is to its parameter,
So is the rectangle of the abscisses,
To the square of their ordinate.

THEOREM III.

As the Square of the Conjugate Axis :
To the Square of the Transverse Axis ::
The Sum of the Squares of the Semi-conjugate, and
Distance of the Centre from any Ordinate of the Axis:
The Square of their Ordinate.

That is,
$ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.



For, draw the ordinate ED to the transverse AB.

Then, by theor. 1. $ca^2 : CA^2 :: DE^2 : AD . DB$ or $CD^2 - CA^2$,
or - - $ca^3 : CA^2 :: cd^2 : dE^2 - CA^2$.
But - - $ca^2 : CA^2 :: ca^2 : CA^2$.
theref. by compos. $ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.
In like manner, $CA^2 : ca^2 :: CA^2 + CD^2 : De^2$. Q. E. D.

Corol. By the last theor. $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$,
and by this theor. $CA^2 : ca^2 :: CD^2 + CA^2 : De^2$,
therefore - $DE^2 : De^2 :: CD^2 - CA^2 : CD^2 + CA^2$.
In like manner, $de^2 : dE^2 :: cd^2 - ca^2 : cd^2 + ca^2$.

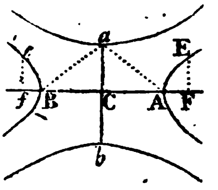
THEOREM

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Sum of the Squares of the Semi-axes.

Or, the Square of the Distance between the Foci, is equal to the Sum of the Squares of the two Axes.

That is,
$CF^2 = CA^2 + Ca^2$, or
$Ff^2 = AB^2 + ab^2$



For, to the focus F draw the ordinate FE; which, by the definition, will be the semi-parameter. Then, by the nature of the curve - $CA^2 : Ca^2 :: CF^2 - CA^2 : FE^2$;
and by the def. of the para. $CA^2 : Ca^2 :: \quad Ca^2 \quad : FE^2$;
therefore - - $Ca^2 = CF^2 - CA^2$;
and by addition, - $CF^2 = CA^2 + Ca^2$;
or, by doubling, - $Ff^2 = AB^2 + ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle CAa; and the distance Aa is = CF the focal distance.

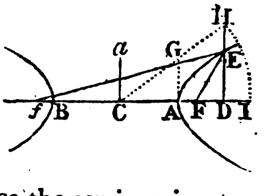
Corol. 2. The conjugate semi-axis Ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For $Ca^2 = CF^2 - CA^2 = \overline{CF + CA} . \overline{CF - CA} = AF . FB$.

THEOREM V.

The Difference of two Lines drawn from the two Foci, to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,
$fE - FE = AB$.



For, draw AG parallel and equal to Ca the semi-conjugate; and join CG, meeting the ordinate DE produced in H; also take CI a 4th proportional to CA, CF, CD.

Then,

Then, by th. 2, $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$;
and, by sim. $\triangle$s, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
consequently $DE^2 = DH^2 - AG^2 = DH^2 - ca^2$.
Also, $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
and, by right-angled triangles, $FE^2 = FD^2 + DE^2$.
therefore $FE^2 = CF^2 - ca^2 - 2CF \cdot CD + CD^2 + DH^2$.
But, by theor. 4, $CF^2 - ca^2 = CA^2$,
and, by supposition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 + DH^2$;
Again, by suppos. $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore - $CI^2 = CD^2 + DH^2 = CH^2$;
consequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.
And the root or side of this square is $FE = CI - CA = AI$.
In the same manner, it is found that $fE = CI + CA = BI$.
Conseq. by subtract. $fE - FE = BI - AI = AB$. Q. E. D.

Corol. 1. Hence $CH = CI$ is a 4th proportional to CA, CF, CD.

Corol. 2. And $fE + FE = 2CH$ or $2CI$; or FE, CH, fE, are in continued arithmetical progression, the common difference being CA the semi-transverse.

Corol. 3. Hence is derived the common method of describing this curve mechanically by points, thus:

In the transverse AB, produced, take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, describe arcs intersecting in E, which will be a point in the curve. In like manner, assuming other points I, as many other points will be found in the curve.

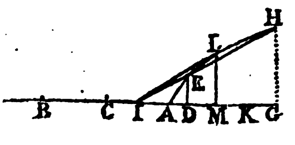
Then, with a steady hand, the curve line may be drawn through all the points of intersection E.

In the same manner are constructed the other two or conjugate hyperbolas, using the axis ab instead of AB.

THEOREM VI.

If from any Point I in the Axis, a Line IL be drawn touching the Curve in one Point L; and the Ordinate LM be drawn; and if C be the Centre or the Middle of AB: Then shall CM be to CI as the Square of AM to the Square of AI.

That is,
$CM : CI :: AM^2 : AI^2$.



For, from the point I draw any line IEH to cut the curve in two points E and H; from which let fall the perps. ED, HG; and bisect DG in K.

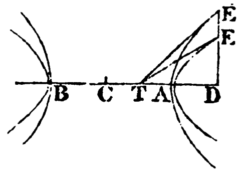
Then, by theor. 1, $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
and by sim. triangles, $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$,
But $DB = CB + CD = CA + CD = CG + CD - AG = 2CK - AG$,
and $GB = CB + CG = CA + CG = CG + CD - AD = 2CK - AD$;
theref. $AD \cdot 2CK - AD \cdot AG : AG \cdot 2CK - AD \cdot AG :: ID^2 : IG^2$,
and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$.
or - $2CK : 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$;
or $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK - AD \cdot AG : ID^2$;
theref. by div. $CK : IK :: AD \cdot AG : AD \cdot 2IK - ID^2$,
and, by div. $CK : CI :: AD \cdot AG : ID^2 - AD \cdot \overline{ID + IA}$,
or - $CK : CI :: AD \cdot AG : AI^2$.

But, when the line IH, by revolving about the point I, comes into the position of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the last proportion becomes $CM : CI :: AM^2 : AI^2$. Q. E. D.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Transverse Axis; the Semi-transverse will be a Mean Proportional between the Distances of the said Two Intersections from the Centre.

That is,
CA is a mean proportional between CD and CT; or CD, CA, CT, are continued proportionals.



For, by th. 6, $CD : CT :: AD^2 : AT^2$,
that is, - $CD : CT :: (CD - CA)^2 : (CA - CT)^2$,
or - - $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$,
and - - $CD : DT :: CD^2 + CA^2 : CD^2 - CT^2$,
or - - $CD : DT :: CD^2 + CA^2 : (CD + CT)\, DT$,
or $CD^2 : CD \cdot DT :: CD^2 + CA^2 : CD \cdot DT + CT \cdot TD$;
hence $CD^2 : CA^2 :: CD \cdot DT : CT \cdot TD$,
and $CD^2 : CA^2 :: CD : CT$,
theref. (th. 78, Geom.) $CD : CA :: CA : CD$. Q. E. D.

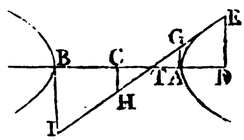
Corol.

Corol. Since CT is always a third proportional to CD, CA; if the points D, A, remain constant, then will the point T be constant also; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every hyperbola described on the same axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from these four Points, namely, the Centre, the two Extremities of the Axis, and the Point of Contact; those Four Perpendiculars will be Proportionals.

That is,
AG : DE :: CH : BI.



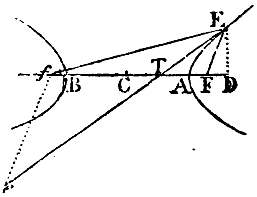
For, by theor. 7, TC : AC :: AC : DC,
theref. by div. TA : AD :: TC : AC or CB,
and by comp. TA : TD :: TC : TB,
and by sim. tri. AG : DE :: CH : BI. Q. E. D.

Corol. Hence TA, TD, TC, TB and TG, TE, TH, TI are also proportionals.. For these are as AG, DE, CH, BI, by similar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; these two Lines will make equal Angles with the Tangent.

That is,
the $\angle$FET = $\angle$fEe.



For, draw the ordinate DE, and fe parallel to FE.
By cor. 1, theor. 5, CA : CD :: CF : CA + FE,
and by th. 7, CA : CD :: CT : CA;

therefore

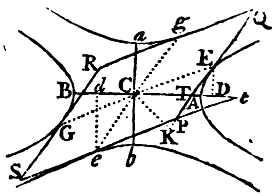
therefore - CT : CF :: CA : CA + FE;
and by add. and sub. TF : Tf :: FE : 2CA + FE or fE by th. 5.
But by sim. tri. TF : Tf : : FE : fE;
therefore - fE = fe, and conseq. ∠e = ∠fEe.
But, because FE is parallel to fe, the ∠e = ∠FET;
therefore the ∠FET = ∠fEe. Q. E. D.

Corol. As opticians find that the angle of incidence is equal to the angle of reflexion, it appears, from this proposition, that rays of light issuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fE is reflected into FE. And this is the reason why the points F, f, are called *foci*, or burning points.

THEOREM X.

All the Parallelograms inscribed between the four Conjugate Hyperbolas are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS =
the rectangle AB . ab.



Let EG, eg be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four less and equal parallelograms. Also, draw the ordinates DE, de, and CK perpendicular to PQ; and let the axis produced meet the sides of the parallelograms, produced, if necessary, in T and t.

Then, by theor. 7, CT : CA :: CA : CD,
and - - ct : CA :: CA : cd;
theref. by equality, CT : ct :: cd : CD;
but, by sim. triangles, CT : ct :: TD : cd,
theref. by equality, TD : cd :: cd : CD,
and the rectangle TD . DC is = the square cd^2.
Again, by theor. 7, CD : CA :: CA : CT,
or, by division, CD : CA :: DA : AT,
and, by composition, CD : DB :: DA : DT;
conseq. the rectangle CD . DT = cd^2 = AD . DB*.

* *Corol.* Because cd^2 = AD . DB = $CD^2 - CA^2$.
therefore $CA^2 = CD^2 - cd^2$.
In like manner $ca^2 = de^2 - DE^2$.

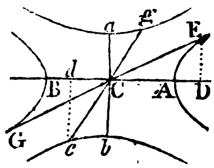
But,

But, by theor. 1, $CA^2 : ca^2 :: (AD \cdot DB \text{ or}) \; cd^2 : DE^2$,
therefore - $CA : ca :: cd : DE$;
In like manner, $CA : ca :: CD : de$;
or - - - $ca : de :: CA : CD$.
But, by theor. 7, $CT : CA :: CA : CD$;
theref. by equality, $CT : CA :: ca : de$.
But, by sim. tri. $CT : CK :: ce : de$;
theref. by equality, $CK : CA :: ca : ce$.
and the rectangle $CK \cdot ce = CA \cdot ca$.
But the rect. $CK \cdot ce =$ the parallelogram $CEPe$,
theref. the rect. $CA \cdot ce =$ the parallelogram $CEPe$,
conseq. the rect. $AB \cdot ab =$ the paral. $PQRS$. Q. E. D.

THEOREM XI.

The Difference of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely the Difference of the Squares of the two Axes.

That is,
$AB^2 - ab^2 = EG^2 - eg^2$;
where EG, eg are any conjugate diameters.

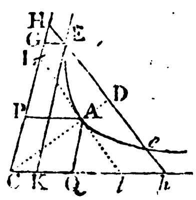


For, draw the ordinates ED, ed.
Then, by cor. to theor. 10, $CA^2 = CD^2 - cd^2$,
and - - - - $ca^2 = de^2 - DE^2$;
theref. the difference $CA^2 - ca^2 = CD^2 + DE^2 - cd^2 - de^2$.
But, by right-angled $\triangle$s, $CE^2 = CD^2 + DE^2$,
and - - - - $ce^2 = cd^2 + de^2$;
theref. the difference $CE^2 - ce^2 = CD^2 + DE^2 - cd^2 - de^2$.
consequently - $CA^2 - ca^2 = CE^2 - ce^2$;
or, by doubling, $AB^2 - ab^2 = EG^2 - eg^2$. Q. E. D.

THEOREM XII.

All the Parallelograms are equal which are formed between the Asymptotes and Curve, by Lines drawn Parallel to the Asymptotes.

That is, the lines GE, EK, AP, AQ, being parallel to the asymptotes CH, cl; then the paral. $CGEK =$ paral. $CPAQ$.



 For,

For, let A be the vertex of the curve, or extremity of the semi-transverse axis AC, perp. to which draw AL or Al, which will be equal to the semi-conjugate, by definition 19. Also, draw HEDeh parallel to Ll,

Then, by theor. 2. $CA^2 : AE^2 :: CD^2 - CA^2 : DE^2$,
and, by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;
theref. by subtract. $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$ or rect. $HE \cdot Eh$;
conseq. the square $AL^2 =$ the rect. $HE \cdot Eh$.

But, by sim. tri. $PA : AL :: GE : EH$,
and, by the same, $QA : Al :: EK : Eh$;
theref. by comp. $PA \cdot AQ : AL^2 :: GE \cdot EK : HE \cdot Eh$;
and, because $AL^2 = HE \cdot Eh$, theref. $PA \cdot AQ = GE \cdot EK$.

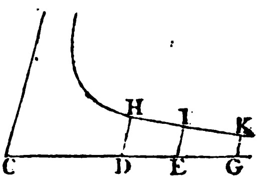
But the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles $GE \cdot EK$ and $PA \cdot AQ$.

Therefore the parallelogram GK = the paral. PQ.

That is, all the inscribed parallelograms are equal to one another. Q. E. D.

Corol. 1. Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or $CG : CP :: PA : GE$. And hence the asymptote continually approaches towards the curve, but never meets it: for GE decreases continually as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GE is infinitely small, or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.

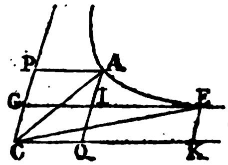
Corol. 2. If the abscisses CD, CE, CG, &c, taken on the one asymptote, be in geometrical progression increasing; then shall the ordinates DH, EI, GK, &c, parallel to the other asymptote, be a decreasing geometrical progression, having the same ratio. For, all the rectangles CDH, CEI, CGK, &c, being equal, the ordinates DH, EI, GK, &c, are reciprocally as the abscisses CD, CE, CG, &c, which are geometricals. And the reciprocals of geometricals are also geometricals, and in the same ratio, but decreasing, or in converse order.



THEOREM

THEOREM XIII.

The three following Spaces, between the Asymptotes and the Curve, are equal; namely, the Sector or Trilinear Space contained by an Arc of the Curve and two Radii, or Lines drawn from its Extremities to the Centre; and each of the two Quadrilaterals, contained by the said Arc, and two Lines drawn from its Extremities parallel to one Asymptote, and the intercepted Part of the other Asymptote.



That is,
The sector CAE = PAEG = QAEK, all standing on the same arc AE.

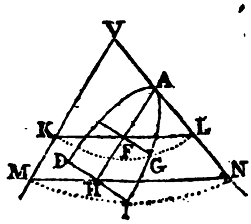
FOR, by theor. 12, CPAQ = CGEK; subtract the common space CGIQ, there remains the paral. PI = the par. IK; to each add the trilineal IAE, then the sum is the quadr. PAEG = QAEK.

Again, from the quadrilateral CAEK take the equal triangles CAQ, CEK, and there remains the sector CAE = QAEK. Therefore CAE = QAEK = PAEG. Q. E. D.

OF THE PARABOLA.

THEOREM I.

The Abscisses are Proportional to the Squares of their Ordinates.



LET AVM be a section through the axis of the cone, and AGIH a parabolic section by a plane perpendicular to the former, and parallel to the side VM of the cone; also let AFH be the common intersection of the two planes, or the axis of the parabola, and FG, HI ordinates perpendicular to it.

Then it will be, as AF : AH :: FG^2 : HI^2.

For, through the ordinates FG, HI draw the circular sections, KGL, MIN, parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the parabola.

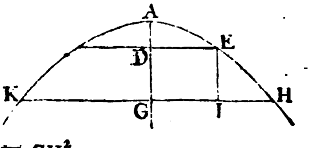
Then, by similar triangles, AF : AH :: FL : HN;
but, because of the parallels, KF = MH;
therefore - - - AF : AH :: KF . FL : MH . HN.
But, by the circle, KF . FL = FG^2, and MH . HN = HI^2;
Therefore - - - AF : AH :: FG^2 : HI^2. Q. E. D.

Corol. Hence the third proportional $\frac{FG^2}{AF}$ or $\frac{HI^2}{AH}$ is a constant quantity, and is equal to the parameter of the axis by defin. 16.

Or AF : FG :: FG : P the parameter.
Or the rectangle P . AF = FG^2.

THEOREM II.

As the Parameter of the Axis :
Is to the Sum of any Two Ordinates ::
So is the Difference of those Ordinates :
To the Difference of their Abscisses :



That is,
P : GH + DE :: GH − DE : DG,
Or, P : KI :: IH : IE.

For, by cor. theor. 1, P . AG = GH^2,
and - - - P . AD = DE^2;
theref. by subtraction, P . DG = $GH^2 - DE^2$.
Or, - - - P . DG = KI . IH,
therefore - - P : KI :: IH : DG or EI. Q. E. D.

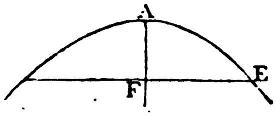
Corol. Hence, because P . EI = KI . IH,
and, by cor. theor. 1, P . AG = GH^2,
therefore - - AG : EI :: GH^2 : KI . IH.

So that any diameter EI is as the rectangle of the segments KI, IH of the double ordinate KH.

THEOREM III.

The Distance from the Vertex to the Focus is equal to $\frac{1}{4}$ of the Parameter, or to Half the Ordinate at the Focus.

That is,
AF $= \frac{1}{2}$FE $= \frac{1}{4}$P,
where F is the focus.

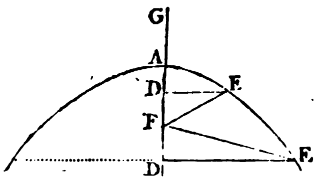


For, the general property is AF : FE :: FE : P.
But, by definition 17, - FE $= \frac{1}{2}$P;
therefore also - - AF $= \frac{1}{2}$FE $= \frac{1}{4}$P. Q. E. D.

THEOREM IV.

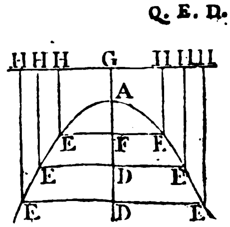
A Line drawn from the Focus to any Point in the Curve, is equal to the Sum of the Focal Distance and the Absciss of the Ordinate to that Point.

That is,
FE $=$ FA $+$ AD $=$ GD,
taking AG $=$ AF.



For, since FD $=$ AD $\backsim$ AF,
theref. by squaring, $FD^2 = AF^2 - 2AF \cdot AD + AD^2$,
But, by cor. theor. 1, $DE^2 = P \cdot AD = 4AF \cdot AD$;
theref. by addition, $FD^2 + DE^2 = AF^2 + 2AF \cdot AD + AD^2$,
But, by right-ang. tri. $FD^2 + DE^2 = FE^2$;
therefore - - $FE^2 = AF^2 + 2AF \cdot AD + AD^2$,
and the root or side is FE $=$ AF $+$ AD,
or - - - FE $=$ GD, by taking AG $=$ AF.
Q. E. D.

Corol. 1. If, through the point G, the line GH be drawn perpendicular to the axis, it is called the directrix of the parabola. The property of which, from this theorem, it appears, is this: That drawing any lines HE parallel to the axis, HE is always equal to FE the distance of the focus from the point E.



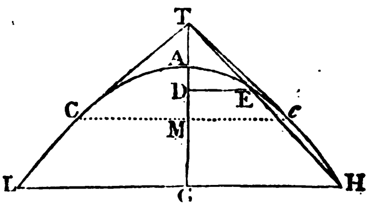
Corol.

Corol. 2. Hence also the curve is easily described by points. Namely, in the axis produced take AG = AF the focal distance, and draw a number of lines EE perpendicular to the axis AD; then with the distances GD, GD, GD, &c, as radii, and the centre F, draw arcs crossing the parallel ordinates in E, E, E, &c. Then draw the curve through all the points E, E, E.

THEOREM V.

If a Tangent be drawn to any Point of the Parabola, meeting the Axis produced; and if an Ordinate to the Axis be drawn from the Point of Contact; then the Abscis of that Ordinate will be equal to the External Part of the Axis.

That is, if TC touch the curve at the point C; then is AT = AM.



For, from the point T, draw any line cutting the curve in the two points E, H: to which draw the ordinates DE, GH; also draw the ordinate MC to the point of contact C.

Then, by th. 1, $AD : AG :: DE^2 : GH^2$;
and, by sim. tri. $TD^2 : TG^2 :: DE^2 : GH^2$;
theref. by equality, $AD : AG :: TD^2 : TG^2$;
and, by division, $AD : DG :: TD^2 : TG^2 - TD^2$ or $DG . (TD + TG)$,
or - - $AD : TD :: TD : TD + TG$;
and, by division, $AD : AT :: TD : TG$,
and again by div. $AD : AT :: AT : AG$;
or - - AT is a mean propor. between AD, AG.

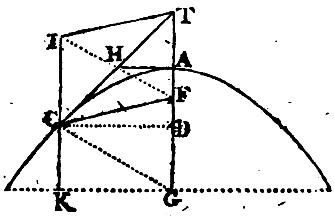
Now, if the line TH be supposed to revolve about the point T; then, as it recedes farther from the axis, the points E and H approach towards each other, the point E descending, and the point H ascending, till at last they meet in the point C, when the line becomes a tangent to the curve at C. And then the points D and G meet in the point M, and the ordinates DE, GH in the ordinate CM. Consequently AD, AG, becoming each equal to AM, their mean proportional AT will be equal to the absciss AM. That is, the external part of the axis, cut off by a tangent, is equal to the absciss of the ordinate to the point of contact. Q. E. D.

THEOREM

THEOREM VI.

If a Tangent to the Curve meet the Axis produced; then the Line drawn from the Focus to the Point of Contact, will be equal to the Distance of the Focus from the Intersection of the Tangent and Axis.

That is,
FC = FT.



FOR, draw the ordinate DC to the point of contact C.

Then, by theor. 5, AT = AD;
therefore - FT = AF + AD.
But, by theor. 4, FC = AF + AD;
theref. by equality, FC = FT. Q. E. D.

Corol. 1. If CG be drawn perpendicular to the curve, or to the tangent, at C; then shall FG = FC = FT.

For, draw FH perpendicular to TC, which will also bisect TC, because FT = FC; and therefore, by the nature of the parallels, FH also bisects TG in F. And consequently FG = FT = FC.

So that F is the centre of a circle passing through T, C, G.

Corol. 2. The tangent at the vertex AH, is a mean proportional between AF and AD.

For, because FHT is a right angle,
therefore - AH is a mean between AF, AT,
or between - AF, AD, because AD = AT.
Likewise, - FH is a mean between FA, FT,
or between FA, FC.

Corol. 3. The tangent TC makes equal angles with FC and the axis FT.

For, because FT = FC,
therefore the ∠ FCT = ∠ FTC.
Also, the angle GCF = the angle GCK,
drawing ICK parallel to the axis AG.

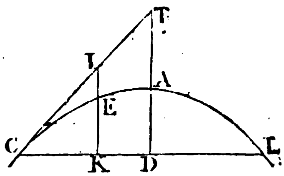
Corol.

Corol. 4. And because the angle of incidence GCK is = the angle of reflection GCF; therefore a ray of light falling on the curve in the direction KC, will be reflected to the focus F. That is, all rays parallel to the axis, are reflected to the focus, or burning point.

THEOREM VII.

If there be any Tangent, and a Double Ordinate drawn from the Point of Contact, and also any Line parallel to the Axis, limited by the Tangent and Double Ordinate: then shall the Curve divide that Line in the same Ratio, as the Line divides the Double Ordinate.

That is,
IE : EK :: CK : KL.

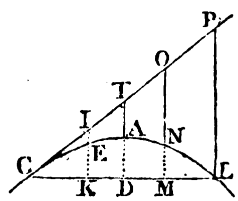


FOR, by sim. triangles,	CK : KI :: CD : DT or 2DA;
but, by the def. the param.	P : CL :: CD : 2DA;
therefore, by equality,	P : CK :: CL : KI.
But, by theor. 2, - -	P : CK :: KL : KE;
therefore, by equality,	CL : KL :: KI : KE;
and, by division, -	CK : KL :: IE : EK. Q. E. D.

THEOREM VIII.

The same being supposed as in theor. 7; then shall the External Part of the Line between the Curve and Tangent, be proportional to the Square of the intercepted Part of the Tangent, or to the Square of the intercepted Part of the Double Ordinate.

That is, IE is as CI^2 or as CK^2.
and IE, TA, ON, PL, &c,
are as CI^2, CT^2, CO^2, CP^2, &c,
or as CK^2, CD^2, CM^2, CL^2, &c.



FOR,

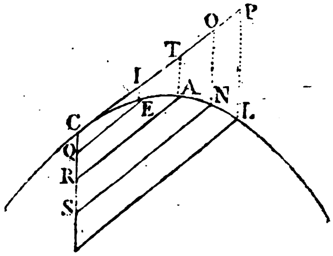
FOR, by theor. 7, IE : EK :: CK : KL,
or, by equality, IE : EK :: CK^2 : CK . KL.
But, by cor. th. 2, EK is as the rect. CK . KL,
therefore - - IE is as CK^2, or as CI^2. Q. E. D.

Corol. As this property is common to every position of the tangent, if the lines IE, TA, ON, &c, be appended on the points I, T, O, &c, and moveable about them, and of such lengths as that their extremities E, A, N, &c, be in the curve of a parabola in some one position of the tangent; then making the tangent revolve about the point C, it appears that the extremities E, A, N, &c, will always form the curve of some parabola, in every position of the tangent.

THEOREM IX.

The Abscisses of any Diameter, are as the Squares of their Ordinates.

That is, CQ, CR, CS, &c, are as QE^2, RA^2, SN^2, &c. Or CQ : CR :: QE^2 : RA^2, &c.



FOR, draw the tangent CT, and the externals EI, AT, NO, &c, parallel to the axis, or to the diameter CS.

Then, because the ordinates QE, RA, SN, &c, are parallel to the tangent CT, by the definition of them, therefore all the figures IQ, TR, OS, &c, are parallelograms, whose opposite sides are equal;

namely, - - - IE, TA, ON, &c,
are equal to - CQ, CR, CS, &c.
Therefore, by theor. 8, CQ, CR, CS, &c,
are as - - CI^2, CT^2, CO^2, &c,
or as their equals - QE^2, RA^2, SN^2, &c. Q. E. D.

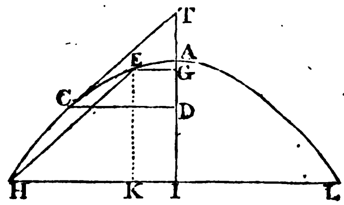
Corol. Here, like as in theor. 2, the difference of the abscisses is as the difference of the squares of their ordinates, or as the rectangles under the sum and difference of the ordinates, the rectangle of the sum and difference of the ordinates

ordinates being equal to the rectangle under the difference of the abscisses and the parameter of that diameter, or a third proportional to any absciss and its ordinate.

THEOREM X.

If a Line be drawn parallel to any Tangent, and cut the Curve in two Points; then if two Ordinates be drawn to the Intersections, and a third to the Point of Contact, these three Ordinates will be in Arithmetical Progression, or the Sum of the Extremes will be equal to Double the Mean.

That is,
EG + HI = 2CD.



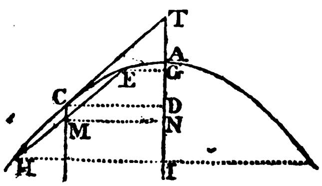
For, draw EK parallel to the axis, and produce HI to L.
Then, by sim. triangles, EK : HK : : TD or 2AD : CD;
but, by theor. 2, - EK : HK : : KL : P the param.
theref. by equality, 2AD : KL : : CD : P.
But, by the defin. 2AD : 2CD : : CD : P;
theref. the 2d terms are equal, KL = 2CD,.
that is, - - EG + HI = 2CD. Q. E. D.

Corol. When the point E is on the other side of AI; then HI — GE = 2CD.

THEOREM XI.

Any Diameter bisects all its Double Ordinates, or Lines parallel to the Tangent at its Vertex.

That is,
ME = MH.



FOR,

FOR, to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.

Then, by theor. 10, 2MN or 2CD = EG + HI, therefore M is the middle of EH.

And, for the same reason, all its parallels are bisected.

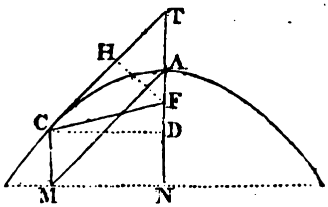
Q. E. D.

SCHOL. Hence, as the abscisses of any diameter and their ordinates have the same relations as those of the axis, namely, that the ordinates are bisected by the diameter, and their squares proportional to the abscisses; so all the other properties of the axis and its ordinates and abscisses, before demonstrated, will likewise hold good for any diameter and its ordinates and abscisses. And also those of the parameters, understanding the parameter of any diameter, as a third proportional to any absciss and its ordinate. Some of the most material of which are demonstrated in the following theorems.

THEOREM XII.

The Parameter of any Diameter is equal to four Times the Line drawn from the Focus to the Vertex of that Diameter.

That is, 4FC = p, the param. of the diam. CM.



FOR, draw the ordinate MA parallel to the tangent CT: also CD, MN perpendicular to the axis AN, and FH perpendicular to the tangent CT.

Then the abscisses AD, CM or AT, being equal, by theor. 5, the parameters will be as the squares of the ordinates CD, MA or CT, by the definition;

that is, - -	P : p :: CD^2 : CT^2,
But, by sim. tri. -	FH : FT :: CD : CT;
therefore - -	P : p :: FH^2 : FT^2.
But, by cor. 2, th. 6,	FH^2 = FA . FT;
therefore - -	P : p :: FA . FT : FT^2.
or, by equality, -	P : p :: FA : FT or FC.

By

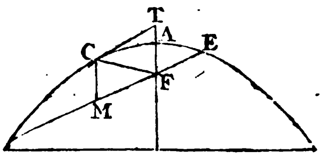
But, by theor. 3,	$P = 4FA$,	
and therefore -	$p = 4FT$ or $4FC$.	Q. E. D.

Corol. Hence the parameter p of the diameter CM is equal to $4FA + 4AD$, or to $P + 4AD$, that is, the parameter of the axis added to $4AD$.

THEOREM XIII.

If an Ordinate to any Diameter, pass through the Focus, it will be equal to Half its Parameter; and its Absciss equal to One Fourth of the same Parameter.

That is, $CM = \frac{1}{4}p$,
and $ME = \frac{1}{2}p$.



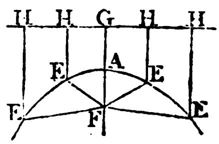
FOR, join FC, and draw the tangent CT.

By the parallels,	$CM = FT$;
and, by theor. 6,	$FC = FT$;
also, by theor. 12,	$FC = \frac{1}{4}p$;
therefore - -	$CM = \frac{1}{4}p$.

Again, by the defin.	CM or $\frac{1}{4}p : ME :: ME : p$,	
and consequently	$ME = \frac{1}{2}p = 2CM$.	Q. E. D.

Corol. 1. Hence, of any diameter, the double ordinate which passes through the focus, is equal to the parameter, or to quadruple its absciss.

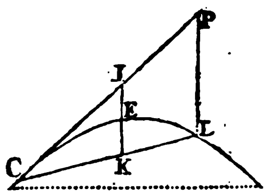
Corol. 2. Hence, and from cor. 1 to theor. 4, and theor. 6 and 12, it appears, that if the directrix GH be drawn, and any lines HE, HE, parallel to the axis; then every parallel HE will be equal to EF, or $\frac{1}{4}$ of the parameter of the diameter to the point E.



THEOREM

THEOREM XIV.

If there be a Tangent, and any Line drawn from the Point of Contact and meeting the Curve in some other Point, as also another Line parallel to the Axis, and limited by the First Line and the Tangent: then shall the Curve divide this Second Line in the same Ratio, as the Second Line divides the First Line.



That is,
IE : EK :: CK : KL.

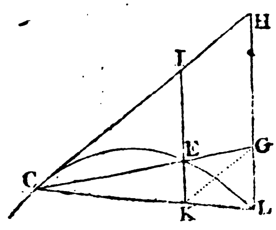
FOR, draw LP parallel to IK, or to the axis.

Then by theor. 8,	IE : PL :: CI^2 : CP^2,
or, by sim. tri. -	IE : PL :: CK^2 : CL^2.
Also, by sim. tri.	IK : PL :: CK : CL,
or - - -	IK : PL :: CK^2 : CK . CL;
therefore by equality,	IE : IK :: CK . CL : CL^2;
or - - - -	IE : IK :: CK : CL;
and, by division,	IE : EK :: CK : KL. Q. E. D.

Corol. When CK = KL, then IE = EK = $\frac{1}{2}$IK.

THEOREM XV.

If from any Point of the Curve there be drawn a Tangent, and also Two Right Lines to cut the Curve; and Diameters be drawn through the Points of Intersection E and L, meeting those Two Right Lines in two other Points G and K: Then will the Line KG joining these last Two Points be parallel to the Tangent.



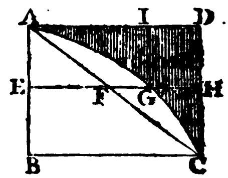
FOR

For, by theor. 14, CK : KL :: EI : EK;
and by composition, CK : CL :: EI : KI;
and by the parallels CK : CL :: GH : LH;
But, by sim. tri. - CK : CL :: KI : LH;
theref. by equal. - KI : LH :: GH : LH:
consequently - KI = GH,
and therefore - KG is parallel and equal to IH. Q. E. D.

THEOREM XVI.

If a Rectangle be described about a Parabola, having the same Base and Altitude; and a diagonal Line be drawn from the Vertex to the Extremity of the Base of the Parabola, forming a right-angled Triangle, of the same Base and Altitude also; then any Line or Ordinate drawn across the three Figures, perpendicular to the Axis, will be cut in Continual Proportion by the Sides of those Figures.

That is,
EF : EG :: EG : EH,
Or, EF, EG, EH, are in continued proportion.



For, by theor. 1, AB : AE :: BC^2 : EG^2,
and, by sim. tri. - AB : AE :: BC : EF,
theref. of equality, - EF : BC :: EG^2 : BC^2,
that is - - - EF : EH :: EG^2 : EH^2,
theref. by Geom. th. 78, EF, EG, EH are proportionals,
or - - - EF : EG :: EG : EH. Q. E. D.

THEOREM XVII.

The Area or Space of a Parabola, is equal to Two-Thirds of its Circumscribing Parallelogram.

That is, the space ABCGA = $\frac{2}{3}$ ABCD;
or, the space ADCGA = $\frac{1}{3}$ ABCD.

For, conceive the space ADCGA to be composed of, or divided into, indefinitively small parts, by lines parallel to DC or AB, such as IG, which divide AD into like small and equal parts, the number or sum of which is expressed by the line AD. Then,

by the parabola, BC^2 : EG^2 :: AB : AE,
that is, - . AD^2 : AI^2 :: DC : IG.

Hence

Hence it follows, that any one of these narrow parts, as IG, is $= \frac{DC}{AD^2} \times AI^2$; hence, AD and DC being given or constant quantities, it appears that the said parts IG, &c, are proportional to AI^2, &c, or proportional to a series of square numbers, whose roots are in arithmetical progression, and the area ADCGA equal to $\frac{DC}{AD^2}$ drawn into the sum of such a series of arithmeticals, the number of which is expressed by AD.

Now, by the remark at pag. 217, vol. i, the sum of the squares of such a series of arithmeticals, is expressed by $\frac{1}{6}n \cdot \overline{n+1} \cdot \overline{2n+1}$, where n denotes the number of them. In the present case, n represents an infinite number, and then the two factors $n + 1$, $2n + 1$, become only n and $2n$, omitting the 1 as inconsiderable in respect of the infinite number n: hence the expression above becomes barely $\frac{1}{6}n \cdot n \cdot 2n = \frac{1}{3}n^3$.

To apply this to the case above: n will denote AD or BC; and the sum of all the AI^2's becomes $\frac{1}{3}AD^3$ or $\frac{1}{3}BC^3$; consequently the sum of all the $\frac{DC}{AD^2} \times AI^2$'s, is $\frac{DC}{AD^2} \times \frac{1}{3}AD^3 = \frac{1}{3}AD \cdot DC = \frac{1}{3}BD$, which is the area of the exterior part ADCGA. That is, the said exterior part ADCGA, is $\frac{1}{3}$ of the parallelogram ABCD; and consequently the interior part ABCGA is $\frac{2}{3}$ of the same parallelogram. Q. E. D.

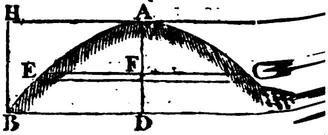
Corol. The part AFCGA, inclosed between the curve and the right line AFC, is $\frac{1}{6}$ of the same parallelogram, being the difference between ABCGA and the triangle ABCFA, that is between $\frac{2}{3}$ and $\frac{1}{2}$ of the parallelogram.

THEOREM XVIII.

The Solid Content of a Paraboloid (or Solid generated by the Rotation of a Parabola about its Axis), is equal to Half its Circumscribing Cylinder.

LET ABC be a paraboloid, generated by the rotation of the parabola AC about its axis AD. Suppose the axis AD be divided into an infinite number of equal parts, through which let circular planes pass, as EFG, all those circles making up the whole solid paraboloid.

Now



Now if c = the number 3·1416, then $2c \times$ FG is the circumference of the circle EFG whose radius is FG; therefore $c \times FG^2$ is the area of that circle.

But, by cor. theor. 1, Parabola, $p \times AF = FG^2$, where p denotes the parameter of the parabola; consequently $pc \times AF$ will also express the same circular section EG, and therefore $pc \times$ the sum of all the AF's will be the sum of all those circular sections, or the whole content of the solid paraboloid.

But all the AF's form an arithmetical progression, beginning at 0 or nothing, and having the greatest term and the sum of all the terms each expressed by the whole axis AD. And since the sum of all the terms of such a progression, is equal to $\frac{1}{2} AD \times AD$ or $\frac{1}{2} AD^2$, half the product of the greatest term and the number of terms; therefore $\frac{1}{2} AD^2$ is equal to the sum of all the AF's, and consequently $pc \times \frac{1}{2} AD^2$, or $\frac{1}{2} c \times p \times AD^2$, is the sum of all the circular sections, or the content of the paraboloid.

But, by the parabola, $p : DC :: DC : AD$, or $p = \frac{DC^2}{AD}$; consequently $\frac{1}{2} c \times p \times AD^2$ becomes $\frac{1}{2} c \times AD \times DC^2$ for the solid content of the paraboloid. But $c \times AD \times DC^2$ is equal to the cylinder BCIH; consequently the paraboloid is the half of its circumscribing cylinder. Q. E. D.

THEOREM XIX.

The Solidity of the Frustum BEGC of the Paraboloid, is equal to a Cylinder whose Height is DF, and its Base Half the Sum of the two Circular Bases EG, BC.

For, by the last theor. $\frac{1}{2} pc \times AD^2 =$ the solid ABC,
and, by the same, $\frac{1}{2} pc \times AF^2 =$ the solid AEG,
theref. the diff. $\frac{1}{2} pc \times (AD^2 - AF^2) =$ the frust. BEGC.
But $AD^2 - AF^2 = DF \times (AD + AF)$,
theref. $\frac{1}{2} pc \times DF \times (AD + AF) =$ the frust. BEGC.
But, by the parab. $p \times AD = DC^2$, and $p \times AF = FG^2$;
theref. $\frac{1}{2} c \times DF \times (DC^2 + FG^2) =$ the frust. BEGC.

Q. E. D.

OF

OF MOTION, FORCES, &c

DEFINITIONS.

Art. 1. BODY is the mass, or quantity of matter, in any material substance; and it is always proportional to its weight or gravity, whatever its figure may be.

2. Body is either Hard, Soft, or Elastic. A Hard Body is that whose parts do not yield to any stroke or percussion, but retains its figure unaltered. A Soft Body is that whose parts yield to any stroke or impression, without restoring themselves again; the figure of the body remaining altered. And an Elastic Body is that whose parts yield to any stroke, but which presently restore themselves again, and the body regains the same figure as before the stroke.

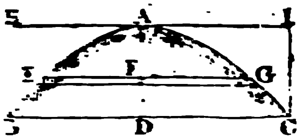
We know of no bodies that are absolutely, or perfectly, either hard, soft, or elastic; but all partaking these properties, more or less, in some intermediate degree.

3. Bodies are also either Solid or Fluid. A Solid Body, is that whose parts are not easily moved among one another, and which retains any figure given to it. But a Fluid Body is that whose parts yield to the slightest impression, being easily moved among one another; and its surface, when left to itself, is always observed to settle in a smooth plane at the top.

4. Density is the proportional weight or quantity of matter in any body. So, in two spheres, or cubes, &c, of equal size or magnitude; if the one weigh only one pound, but the other 2 pounds; then the density of the latter is double the density of the former; if it weigh 3 pounds, its density is triple; and so on.

5. Motion is a continual and successive change of place.— If the body move equally, or pass over equal spaces in equal times, it is called Equable or Uniform Motion. But if it increase or decrease, it is Variable Motion; and it is called Accelerated Motion in the former case, and Retarded Motion in the latter.—Also, when the moving body is considered

[illegible] = the number [illegible]



[illegible] Parabola, $p \times$ AF = FG^2, where p [illegible] consequently $p c \times$ AF [illegible] and therefore [illegible] of all those [illegible] of the solid parabola.

[illegible] progression, beginning [illegible] the greatest term and the [illegible] the whole axis AD. [illegible] of such a progression, is [illegible] half the product of the greatest [illegible] therefore $\frac{1}{2}$ AD^2 is equal to [illegible] $\frac{1}{2}$ AD^2, or $\frac{1}{2}$ c [illegible] sections, or the [illegible]

[illegible] $= \frac{DC^2}{AD}$; con[illegible] becomes [illegible] $\times$ AD $\times$ DC^2 for the [illegible] $\times$ AD $\times$ DC^2 is equ[illegible] the paraboloid is the h[illegible] [illegible]

Q. E. D.

THEOREM [illegible]

The [illegible] of the Paraboloid, [illegible] Height is DF, and its Base [illegible] Bases EG, SC.

[illegible] = the solid [illegible]

[illegible] $\times$ AF^2 = the solid

[illegible] $(AD^2 - AF^2)$

But [illegible] $-$ [illegible] $=$ DF $\times$ (AD + A[illegible]

[illegible] (AD + AF) = th[illegible]

[illegible] $\times$ AD = DC^2, and

[illegible] $(DC^2 + FG^2)$ = the

with respect to some other body at rest, it is said to be Absolute Motion. But when compared with others in motion, it is called Relative Motion.

6. Velocity, or Celerity, is an affection of motion, by which a body passes over a certain space in a certain time. Thus, if a body in motion pass uniformly over 40 feet in 4 seconds of time, it is said to move with the velocity of 10 feet per second; and so on.

7. Momentum, or Quantity of Motion, is the power or force in moving bodies, by which they continually tend from their present places, or with which they strike any obstacle that opposes their motion.

8. Force is a power exerted on a body to move it, or to stop it. If the force act constantly, or incessantly, it is a Permanent Force: like pressure or the force of gravity. But if it act instantaneously, or but for an imperceptibly small time, it is called Impulse, or Percussion: like the smart blow of a hammer.

9. Forces are also distinguished into Motive, and Accelerative or Retarding. A Motive or Moving Force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in any body, when acting, either by percussion, or for a certain time as a permanent force.

10. Accelerative, or Retardive Force, is commonly understood to be that which affects the velocity only: or it is that by which the velocity is accelerated or retarded; and it is equal or proportional to the motive force directly, and to the mass or body moved inversely.—So, if a body of 2 pounds weight, be acted on by a motive force of 40; then the accelerating force is 20. But if the same force of 40 act on another body of 4 pounds weight; then the accelerating force in this latter case is only 10; and so is but half the former, and will produce only half the velocity.

11. Gravity, or Weight, is that force by which a body endeavours to fall downwards. It is called Absolute Gravity, when the body is in empty space; and Relative Gravity, when immersed in a fluid.

12. Specific Gravity is the proportion of the weights of different bodies of equal magnitude; and so is proportional to the density of the body.

AXIOMS.

AXIOMS.

13. EVERY body naturally endeavours to continue in its present state, whether it be at rest, or moving uniformly in a right line.

14. The Change or Alteration of Motion, by any external force, is always proportional to that force, and in the direction of the right line in which it acts.

15. Action and Re-action, between any two bodies, are equal and contrary. That is, by Action and Re-action, equal changes of motion are produced in bodies acting on each other; and these changes are directed towards opposite or contrary parts.

GENERAL LAWS OF MOTION, &c.

PROPOSITION I.

16. *The Quantity of Matter, in all Bodies, is in the Compound Ratio of their Magnitudes and Densities.*

THAT is, b is as md; where b denotes the body or quantity of matter, m its magnitude, and d its density.

For, by art. 4, in bodies of equal magnitude, the mass or quantity of matter is as the density. But, the densities remaining, the mass is as the magnitude: that is, a double magnitude contains a double quantity of matter, a triple magnitude a triple quantity, and so on. Therefore the mass is in the compound ratio of the magnitude and density.

17. *Corol.* 1. In similar bodies, the masses are as the densities and cubes of the diameters, or of any like linear dimensions.—For the magnitudes of bodies are as the cubes of the diameters, &c.

18. *Corol.* 2. The masses are as the magnitudes and specific gravities.—For, by art. 4 and 12, the densities of bodies are as the specific gravities.

19. *Scholium.* Hence, if b denote any body, or the quantity of matter in it, m its magnitude, d its density, g its

specific gravity, and a its diameter or other dimension; then, $\propto$ (pronounced or named *as*) being the mark for general proportion, from this proposition and its corollaries we have these general proportions:

$$b \propto md \propto mg \propto a^3 d,$$

$$m \propto \frac{b}{d} \propto \frac{b}{g} \propto a^3,$$

$$d \propto \frac{b}{m} \propto g \propto \frac{mg}{a^3},$$

$$a^3 \propto \frac{b}{d} \propto m \propto \frac{mg}{d}.$$

PROPOSITION II.

20. *The Momentum, or Quantity of Motion, generated by a Single Impulse, or any Momentary Force, is as the Generating Force.*

THAT is, m is as f; where m denotes the momentum, and f the force.

For every effect is proportional to its adequate cause. So that a double force will impress a double quantity of motion; a triple force, a triple motion; and so on. That is, the motion impressed, is as the motive force which produces it.

PROPOSITION III.

21. *The Momenta, or Quantities of Motion, in Moving Bodies, are in the Compound Ratio of the Masses and Velocities.*

That is, m is as bv.

FOR, the motion of any body being made up of the motions of all its parts, if the velocities be equal, the momenta will be as the masses; for a double mass will strike with a double force; a triple mass, with a triple force; and so on. Again, when the mass is the same, it will require a double force to move it with a double velocity, a triple force with a triple velocity, and so on; that is, the motive force is as the velocity; but the momentum impressed, is as the force which produces it, by prop. 2; and therefore the momentum is as the velocity when the mass is the same. But the momentum was found to be as the mass when the velocity is the same. Consequently,

Consequently, when neither are the same, the momentum is in the compound ratio of both the mass and velocity.

PROPOSITION IV.

22. *In Uniform Motions, the Spaces described are in the Compound Ratio of the Velocities and the Times of their Description.*

That is, s is as tv.

FOR, by the nature of uniform motion, the greater the velocity, the greater is the space described in any one and the same time; that is, the space is as the velocity, when the times are equal. And when the velocity is the same, the space will be as the time; that is, in a double time a double space will be described; in a triple time, a triple space; and so on. Therefore universally, the space is in the compound ratio of the velocity, and the time of description.

23. *Corol.* 1. In uniform motions, the time is as the space directly, and velocity reciprocally; or as the space divided by the velocity. And when the velocity is the same, the time is as the space. But when the space is the same, the time is reciprocally as the velocity.

24. *Corol.* 2. The velocity is as the space directly and the time reciprocally; or as the space divided by the time. And when the time is the same, the velocity is as the space. But when the space is the same, the velocity is reciprocally as the time.

Scholium.

25. In uniform motions generated by momentary impulse, let $b =$ any body or quantity of matter to be moved,

$f =$ force of impulse acting on the body b,
$v =$ the uniform velocity generated in b,
$m =$ the momentum generated in b,
$s =$ the space described by the body b,
$t =$ the time of describing the space s with the veloc. v.

Then from the last three propositions and corollaries, we have these three general proportions, namely, $f \propto m$, $m \propto bv$, and $s \propto tv$; from which is derived the following table of the general relations of those six quantities, in uniform motions, and impulsive or percussive forces:

$f \propto$

$$f \propto m \propto bv \propto \frac{bs}{t}.$$

$$m \propto f \propto bv \propto \frac{bs}{t}.$$

$$b \propto \frac{f}{v} \propto \frac{m}{v} \propto \frac{ft}{s} \propto \frac{mt}{s}.$$

$$s \propto tv \propto \frac{ft}{b} \propto \frac{tm}{b}.$$

$$v \propto \frac{s}{t} \propto \frac{f}{b} \propto \frac{m}{b}.$$

$$t \propto \frac{s}{v} \propto \frac{bs}{f} \propto \frac{bs}{m}.$$

By means of which, may be resolved all questions relating to uniform motions, and the effects of momentary or impulsive forces.

PROPOSITION V.

26. *The Momentum generated by a Constant and Uniform Force, acting for any Time, is in the Compound Ratio of the Force and Time of Acting.*

That is, m is as ft.

FOR, supposing the time divided into very small parts, by prop. 2, the momentum in each particle of time is the same, and therefore the whole momentum will be as the whole time, or sum of all the small parts. But by the same prop. the momentum for each small time, is also as the motive force. Consequently the whole momentum generated, is in the compound ratio of the force and time of acting.

27. *Corol.* 1. The motion, or momentum, lost or destroyed in any time, is also in the compound ratio of the force and time. For whatever momentum any force generates in a given time; the same momentum will an equal force destroy in the same or equal time; acting in a contrary direction.

And the same is true of the increase or decrease of motion, by forces that conspire with, or oppose the motion of bodies.

[illegible]*l.* 2. The velocity generated, or destroyed, in any [illegible]ctly as the force and time, and reciprocally as [illegible]mass of matter.—For, by this and the 3d prop. [illegible] ratio of the body and velocity, is as that of [illegible]me; and therefore the velocity is as the force [illegible] by the body. And if the body and force [illegible]it, the velocity will be as the time.

PROPOSITION

PROPOSITION VI.

29. *The Spaces passed over by Bodies, urged by any Constant and Uniform Forces, acting during any Times, are in the Compound Ratio of the Forces and Squares of the Times directly, and the Body or Mass reciprocally.*

Or, the Spaces are as the Squares of the Times, when the Force and Body are given.

THAT is, s is as $\frac{ft^2}{b}$, or as t^2 when f and b are given. For, let v denote the velocity acquired at the end of any time t, by any given body b, when it has passed over the space s. Then, because the velocity is as the time, by the last corol. therefore $\frac{1}{2}v$ is the velocity at $\frac{1}{2}t$, or at the middle point of the time; and as the increase of velocity is uniform, the same space s will be described in the same time t, by the velocity $\frac{1}{2}v$ uniformly continued from beginning to end. But, in uniform motions, the space is in the compound ratio of the time and velocity; therefore s is as $\frac{1}{2}tv$, or indeed $s = \frac{1}{2}tv$. But, by the last corol. the velocity v is as $\frac{ft}{b}$, or as the force and time directly, and as the body reciprocally. Therefore s, or $\frac{1}{2}tv$, is as $\frac{ft^2}{b}$; that is, the space is as the force and square of the time directly, and as the body reciprocally. Or s is as t^2, the square of the time only, when b and f are given.

30. *Corol.* 1. The space s is also as tv, or in the compound ratio of the time and velocity; b and f being given. For, $s = \frac{1}{2}tv$ is the space actually described. But tv is the space which might be described in the same time t, with the last velocity v, if it were uniformly continued for the same or an equal time. Therefore the space s, or $\frac{1}{2}tv$, which is actually described, is just half the space tv, which would be described with the last or greatest velocity, uniformly continued for an equal time t.

31. *Corol.* 2. The space s is also as v^2, the square of the velocity; because the velocity v is as the time t.

Scholium.

32. Propositions 3, 4, 5, 6, give theorems for resolving all questions relating to motions uniformly accelerated. Thus, put

put b = any body or quantity of matter,
f = the force constantly acting on it,
t = the time of its acting,
v = the velocity generated in the time t,
s = the space described in that time,
m = the momentum at the end of the time.

Then, from these fundamental relations, $m \propto bv$, $m \propto ft$, $s \propto tv$, and $v \propto \frac{ft}{b}$, we obtain the following table of the general relations of uniformly accelerated motions:

$$m \propto bv \propto ft \propto \frac{bs}{t} \propto \frac{fs}{v} \propto \frac{ft^2v}{s} \propto \sqrt{bfs} \quad \sqrt{bftv}.$$

$$b \propto \frac{m}{v} \propto \frac{ft}{v} \propto \frac{mt}{s} \propto \frac{ft^2}{s} \propto \frac{f^2t^3}{ms} \propto \frac{m^2}{fs} \propto \frac{m^2}{ftv} \propto \frac{fs}{v^2}.$$

$$f \propto \frac{m}{t} \propto \frac{bv}{t} \propto \frac{mv}{s} \propto \frac{ms}{t^2v} \propto \frac{m^2}{bs} \propto \frac{m^2}{btv} \propto \frac{bv^2}{s} \propto \frac{bs}{t^2}.$$

$$v \propto \frac{s}{t} \propto \frac{ft}{b} \propto \frac{m}{b} \propto \frac{ms}{ft^2} \propto \frac{fs}{m} \propto \frac{m^2}{bft} \propto \sqrt{\frac{fs}{b}} \propto \frac{f^2st}{m^2}.$$

$$s \propto tv \propto \frac{ft^2}{b} \propto \frac{mt}{b} \propto \frac{ft^2v}{m} \propto \frac{mv}{f} \propto \frac{m^2}{bf} \propto \frac{bv^2}{f} \propto \frac{m^2v}{f^2t}.$$

$$t \propto \frac{s}{v} \propto \frac{m}{f} \propto \frac{bv}{f} \propto \frac{bs}{m} \propto \sqrt{\frac{bs}{f}} \propto \sqrt{\frac{ms}{fv}} \propto \frac{m^2}{bfv}, \text{ \&c.}$$

33. And from these proportions those quantities are to be left out which are given, or which are proportional to each other. Thus, if the body or quantity of matter be always the same, then the space described is as the force and square of the time. And if the body be proportional to the force, as all bodies are in respect to their gravity; then the space described is as the square of the time, or square of the velocity; and in this case, if F be put $= \frac{f}{b}$, the accelerating force; then will

$$s \propto tv \propto \text{F}t^2 \propto \frac{v^2}{\text{F}}.$$

$$v \propto \frac{s}{t} \propto \text{F}t \propto \sqrt{\text{F}s}.$$

$$t \propto \frac{s}{v} \propto \frac{v}{\text{F}} \propto \sqrt{\text{F}}.$$

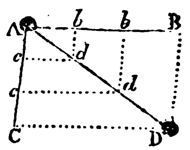
THE

THE COMPOSITION AND RESOLUTION OF FORCES.

34. COMPOSITION of FORCES, is the uniting of two or more forces into one, which shall have the same effect; or the finding of one force that shall be equal to several others taken together, in any different directions. And the Resolution of Forces, is the finding of two or more forces which, acting in any different directions, shall have the same effect as any given single force.

PROPOSITION VII.

35. *If a Body at* A *be urged in the Directions* AB *and* AC, *by any two Similar Forces, such that they would separately cause the Body to pass over the Spaces* AB, AC, *in an equal Time; then if both Forces act together, they will cause the Body to move, in the same Time, through* AD *the Diagonal of the Parallelogram* ABCD.



DRAW *cd* parallel to AB, and *bd* parallel to AC. And while the body is carried over A*b* or *cd* by the force in that direction, let it be carried over *bd* by the force in that direction; by which means it will be found at *d*. Now, if the forces be impulsive or momentary, the motions will be uniform, and the spaces described will be as the times of description:

theref. A*b* or *cd* : AB or CD :: time in A*b* : time in AB,
and *bd* or A*c* : BD or AC :: time in A*c* : time in AC;
but the time in A*b* = time in A*c*, and the time in AB = time in AC; therefore A*b* : *bd* :: AB : BD by equality: hence the point *d* is in the diagonal AD.

And as this is always the case in every point *d*, *d*, &c, therefore the path of the body is the straight line A*d*D, or the diagonal of the parallelogram.

But if the similar forces, by means of which the body is moved in the directions AB, AC, be uniformly accelerating ones, then the spaces will be as the squares of the times; in which case, call the time in *bd* or *cd*, t, and the time in AB or AC, T; then

it will be A*b* or *cd* : AB or CD :: $t^2 : T^2$,
and - *bd* or A*c* : BD or AC :: $t^2 : T^2$,
theref. by equality, A*b* : *bd* :: AB : BD;

and so the body is always found in the diagonal, as before.

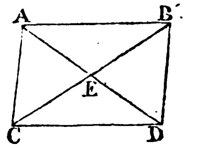
36. Cor

36. *Corol.* 1. If the forces be not similar, by which the body is urged in the directions AB, AC, it will move in some curved line, depending on the nature of the forces.

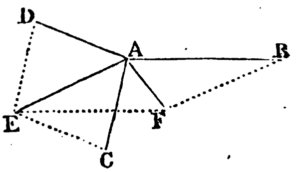
37. *Corol.* 2. Hence it appears, that the body moves over the diagonal AD, by the compound motion, in the very same time that it would move over the side AB, by the single force impressed in that direction, or that it would move over the side AC by the force impressed in that direction.

38. *Corol.* 3. The forces in the directions AB, AC, AD, are respectively proportional to the lines AB, AC, AD, and in these directions.

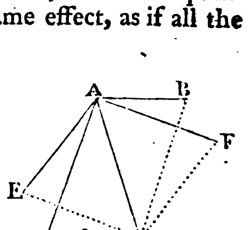
39. *Corol.* 4. The two oblique forces AB, AC, are equivalent to the single direct force AD, which may be compounded of these two, by drawing the diagonal of the parallelogram. Or they are equivalent to the double of AE, drawn to the middle of the line BC. And thus any force may be compounded of two or more other forces; which is the meaning of the expression *composition of forces.*



40. *Exam.* Suppose it were required to compound the three forces AB, AC, AD; or to find the direction and quantity of one single force, which shall be equivalent to, and have the same effect, as if a body A were acted on by three forces in the directions AB, AC, AD, and proportional to these three lines. First reduce the two AC, AD to one AE, by completing the parallelogram ADEC. Then reduce the two AE, AB to one AF by the parallelogram AEFB. So shall the single force AF be the direction, and as the quantity, which shall of itself produce the same effect, as if all the three AB, AC, AD acted together.

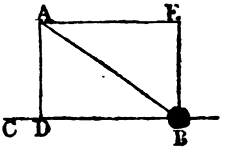


41. *Corol.* 5. Hence also any single direct force AD, may be resolved into two oblique forces, whose quantities and directions are AB, AC, having the same effect, by describing any parallelogram whose diagonal may be AD: and this is called the resolution of forces. So the force AD may be resolved into the two AB, AC, by the parallelogram ABDC



ABDC; or into the two AE, AF, by the parallelogram AEDF; and so on, for any other two. And each of these may be resolved again into as many others as we please.

42. *Corol.* 6. Hence too may be found the effect of any given force, in any other direction, besides that of the line in which it acts; as, of the force AB in any other given direction CB. For draw AD perpendicular to CB; then shall DB be the effect of the force AB in the direction CB. For, the given force AB is equivalent to the two AD, DB, or AE; of which the former AD, or EB, being perpendicular, does not alter the velocity in the direction CB; and therefore DB is the whole effect of AB in the direction CB. That is, a direct force expressed by the line DB acting in the direction DB, will produce the same effect or motion in a body B, in that direction, as the oblique force expressed by, and acting in, the direction AB, produces in the same direction CB. And hence any given force AB, is to its effect in DB, as AB to DB, or as radius to the cosine of the angle ABD of inclination of those directions. For the same reason, the force or effect in the direction AB, is to the force or effect in the direction AD or EB, as AB to AD; or as radius to sine of the same angle ABD, or cosine of the angle DAB of those directions.



43. *Corol.* 7. Hence also, if the two given forces, to be compounded, act in the same line, either both the same way, or the one directly opposite to the other; then their joint or compounded force will act in the same line also, and will be equal to the sum of the two when they act the same way, or to the difference of them when they act in opposite directions; and the compound force, whether it be the sum or difference, will always act in the direction of the greater of the two.

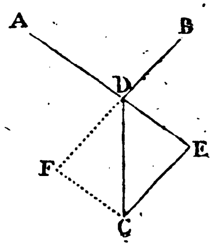
PROPOSITION VIII.

44. *If Three Forces* A, B, C, *acting all together in the same Plane, keep one another in Equilibrio; they will be Proportional to the Three Sides* DE, EC, CD, *of a Triangle, which are drawn Parallel to the Directions of the Forces* AD, DB, CD.

PRODUCE AD, BD, and draw CF, CE parallel to them.

Then

Then the force in CD is equivalent to the two AD, BD, by the supposition; but the force CD is also equivalent to the two ED and CE or FD; therefore, if CD represent the force C, then ED will represent its opposite force A, and CE, or FD, its opposite force B. Consequently the three forces A, B, C, are proportional to DE, CE, CD, the three lines parallel to the directions in which they act.

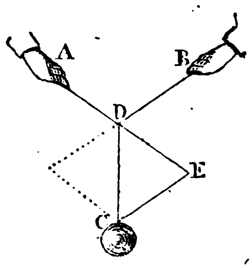


45. *Corol.* 1. Because the three sides CD, CE, DE, are proportional to the sines of their opposite angles E, D, C; therefore the three forces, when in equilibrio, are proportional to the sines of the angles of the triangle made of their lines of direction; namely, each force proportional to the sine of the angle made by the directions of the other two.

46. *Corol.* 2. The three forces, acting against, and keeping one another in equilibrio, are also proportional to the sides of any other triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with those directions. For such a triangle is always similar to the former, which is made by drawing lines parallel to the directions; and therefore their sides are in the same proportion to one another.

47. *Corol.* 3. If any number of forces be kept in equilibrio by their actions against one another; they may be all reduced to two equal and opposite ones.—For, by cor. 4, prop. 7, any two of the forces may be reduced to one force acting in the same plane; then this last force and another may likewise be reduced to another force acting in their plane: and so on, till at last they be all reduced to the action of only two opposite forces; which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.

48. *Corol.* 4. If one of the forces, as C, be a weight, which is sustained by two strings drawing in the directions DA, DB: then the force or tension of the string AD, is to the weight C, or tension of the string DC, as DE to DC; and the force or tension of the other string BD, is to the weight C, or tension of CD, as CE to CD.



49. Corol.

49. *Corol.* 5. If three forces be in equilibrio by their mutual actions; the line of direction of each force, as DC, passes through the opposite angle C of the parallelogram formed by the directions of the other two forces.

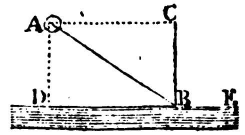
50. *Remark.* These properties, in this proposition and its corollaries, hold true of all similar forces whatever, whether they be instantaneous or continual, or whether they act by percussion, drawing, pushing, pressing, or weighing; and are of the utmost importance in mechanics and the doctrine of forces.

ON THE COLLISION OF BODIES.

PROPOSITION IX.

51. *If a Body strike or act Obliquely on a Plain Surface, the Force or Energy of the Stroke, or Action, is as the Sine of the Angle of Incidence.*

Or, the Force on the Surface is to the same if it had acted Perpendicularly, as the Sine of Incidence is to Radius.



LET AB express the direction and the absolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the sine of the angle ABD, or as AB to AD or BC, drawing AD and BC perpendicular, and AC parallel to DE.

For, by prop. 7, the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the plane, because it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB, as the sine of the angle BAC, or ABD, to radius.

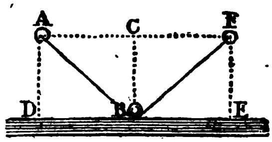
52. *Corol.* 1. If a body act on another, in any direction, and by any kind of force, the action of that force on the second body, is made only in a direction perpendicular to the surface on which it acts. For the force in AB acts on DE only by the force CB, and in that direction.

53. *Corol.* 2. If the plane DE be not absolutely fixed, it will move, after the stroke, in the direction perpendicular to its surface. For it is in that direction that the force is exerted.

PROPOSITION

PROPOSITION XIII.

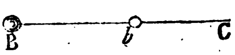
62. *If an Elastic Body* A *impinge on a Firm Plane* DE *at the Point* B, *it will rebound from it in an Angle equal to that in which it struck it; or the Angle of Incidence will be equal to the Angle of Reflexion; namely, the Angle* ABD *equal to the Angle* FBE.



LET AB express the force of the body A in the direction AB; which let be resolved into the two AC, CB, parallel and perpendicular to the plane.—Take BE and CF equal to AC, and draw BF. Now action and reaction being equal, the plane will resist the direct force CB by another BC equal to it, and in a contrary direction; whereas the other AC, being parallel to the plane, is not acted on or diminished by it, but still continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF by composition. But, because AC is equal to BE or CF, and that BC is common, the two triangles BCA, BCF are mutually similar and equal; and consequently the angles at A and F are equal, as also their equal alternate angles ABD, FBE, which are the angles of incidence and reflexion.

PROPOSITION XIV.

63. *To determine the Motion of Non-elastic Bodies, when they strike each other Directly, or in the Same Line of Direction.*



LET the non-elastic body B, moving with the velocity V in the direction Bb, and the body b with the velocity v, strike each other. Then, because the momentum of any moving body is as the mass into the velocity, BV = M is the momentum of the body B, and $bv = m$ the momentum of the body b, which let be the less powerful of the two motions. Then, by prop. 10, the bodies will both move together as one mass in the direction BC after the stroke, whether before the stroke the body b moved towards C or towards B. Now, according as that motion of b was from or towards B, that is, whether the motions were in the same or contrary ways, the momentum after the stroke, in direction BC, will be

be the sum or difference of the momentums before the stroke; namely, the momentum in direction BC will be

BV $+ bv$, if the bodies moved the same way, or
BV $- bv$, if they moved contrary ways, and
BV only, if the body b were at rest.

Then divide each momentum by the common mass of matter $B + b$, and the quotient will be the common velocity after the stroke in the direction BC; namely, the common velocity will be, in the first case

$$\frac{BV + bv}{B + b}, \text{ in the 2d } \frac{BV - bv}{B + b}, \text{ and in the 3d } \frac{BV}{B + b}.$$

64. For example, if the bodies, or weights, B and b, be as 5 to 3, and their velocities V and v, as 6 to 4, or as 3 to 2, before the stroke; then 15 and 6 will be as their momentums, and 8 the sum of their weights; consequently, after the stroke, the common velocity will be as

$$\frac{15 + 6}{8} = \frac{21}{8} \text{ or } 2\tfrac{5}{8} \text{ in the first case,}$$

$$\frac{15 - 6}{8} = \frac{9}{8} \text{ or } 1\tfrac{1}{8} \text{ in the second, and}$$

$$\frac{15}{8} \text{ - - - - - or } 1\tfrac{7}{8} \text{ in the third.}$$

PROPOSITION XV.

65. *If two Perfectly Elastic Bodies impinge on one another: their Relative Velocity will be the same both Before and After the Impulse: that is, they will recede from each other with the Same Velocity with which they approached and met.*

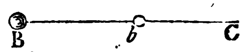
For the compressing force is as the intensity of the stroke; which, in given bodies, is as the relative velocity with which they meet or strike. But perfectly elastic bodies restore themselves to their former figure, by the same force by which they were compressed; that is, the restoring force is equal to the compressing force, or to the force with which the bodies approach each other before the impulse. But the bodies are impelled from each other by this restoring force; and therefore this force, acting on the same bodies, will produce a relative velocity equal to that which they had before: or it will make the bodies recede from each other with the

same velocity with which they before approached, or so as to be equally distant from one another at equal times before and after the impact.

66. *Remark.* It is not meant by this proposition, that each body will have the same velocity after the impulse as it had before; for that will be varied according to the relation of the masses of the two bodies; but that the velocity of the one will be, after the stroke, as much increased as that of the other is decreased, in one and the same direction. So, if the elastic body B move with a velocity V, and overtake the elastic body b moving the same way with the velocity v; then their relative velocity, or that with which they strike, is $V - v$, and it is with this same velocity that they separate from each other after the stroke. But if they meet each other, or the body b move contrary to the body B; then they meet and strike with the velocity $V + v$, and it is with the same velocity that they separate and recede from each other after the stroke. But whether they move forward or backward after the impulse, and with what particular velocities, are circumstances that depend on the various masses and velocities of the bodies before the stroke, and which make the subject of the next proposition.

PROPOSITION XVI.

67. *To determine the Motions of Elastic Bodies after Striking each other directly.*



Let the elastic body B move in the direction BC, with the velocity V; and let the velocity of the other body b be v in the same line; which latter velocity v will be positive if b move the same way as B, but negative if b move in the opposite direction to B. Then their relative velocity in the direction BC is $V - v$; also the momenta before the stroke are BV and bv, the sum of which is $BV + bv$ in the direction BC.

Again, put x for the velocity of B, and y for that of b, in the same direction BC, after the stroke; then their relative velocity is $y - x$, and the sum of their momenta $Bx + by$ in the same direction.

But the momenta before and after the collision, estimated in the same direction, are equal, by prop. 10, as also the relative velocities, by the last prop. Whence arise these two equations:

viz.

viz. $BV + bv = Bx + by$,
and $V - v = y - x$;
the resolution of which equations gives

$$x = \frac{(B - b)V + 2bv}{B + b}, \text{ the velocity of } B,$$

$$y = \frac{-(B - b)v + 2BV}{B + b}, \text{ the velocity of } b,$$

both in the direction BC, when V and v are both positive, or the bodies both moved towards C before the collision. But if v be negative, or the body b moved in the contrary direction before collision, or towards B; then, changing the sign of v, the same theorems become

$$x = \frac{(B - b)V - 2bv}{B + b}, \text{ the velocity of } B,$$

$$y = \frac{(B - b)v + 2BV}{B + b}, \text{ the veloc. of } b, \text{ in the direction BC.}$$

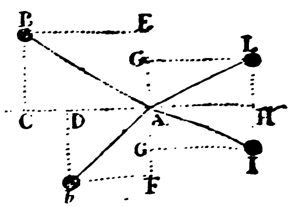
And if b were at rest before the impact, making its velocity $v = 0$, the same theorems give

$$x = \frac{B - b}{B + b}V, \text{ and } y = \frac{2B}{B + b}V, \text{ the velocities in this case.}$$

And, in this case, if the two bodies B and b be equal to each other; then $B - b = 0$, and $\frac{2B}{B + b} = \frac{2B}{2B} = 1$; which give $x = 0$, and $y = V$; that is, the body B will stand still, and the other body b will move on with the whole velocity of the former; a thing which we sometimes see happen in playing at billiards; and which would happen much oftener if the balls were perfectly elastic.

PROPOSITION XVII.

68. *If Bodies strike one another Obliquely, it is proposed to determine their Motions after the Stroke.*



LET the two bodies B, b, move in the oblique directions BA, bA, and strike each other at A, with velocities which are in proportion to the lines BA, bA; to find their motions after the impact. Let CAH represent the plane in which the bodies touch in the point of concourse; to which draw the perpendiculars BC, bD, and complete the rectangles CE, DF. Then the motion in BA is re-

 solved

solved into the two BC, CA; and the motion in *b*A is resolved into the two *b*D, DA; of which the antecedents BC, *b*D, are the velocities with which they directly meet, and the consequents CA, DA, are parallel; therefore by these the bodies do not impinge on each other, and consequently the motions, according to these directions, will not be changed by the impulse; so that the velocities with which the bodies meet, are as BC and *b*D, or their equals EA and FA. The motions therefore of the bodies B, *b*, directly striking each other with the velocities EA, FA, will be determined by prop. 16 or 14, according as the bodies are elastic or non-elastic; which being done, let AG be the velocity, so determined, of one of them, as A; and since there remains also in the body a force of moving in the direction parallel to BE, with a velocity as BE, make AH equal to BE, and complete the rectangle GH: then the two motions in AH and AG, or HI, are compounded into the diagonal AI, which therefore will be the path and velocity of the body B after the stroke. And after the same manner is the motion of the other body *b* determined after the impact.

If the elasticity of the bodies be imperfect in any given degree, then the quantity of the corresponding lines must be diminished in the same proportion.

THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

PROPOSITION XVIII.

69. *All the Properties of Motion delivered in Proposition VI, its Corollaries and Scholium, for Constant Forces, are true in the Motions of Bodies freely descending by their own Gravity; namely, that the Velocities are as the Times, and the Spaces as the Squares of the Times, or as the Squares of the Velocities.*

FOR, since the force of gravity is uniform, and constantly the same, at all places near the earth's surface, or at nearly the same distance from the centre of the earth; and since this is the force by which bodies descend to the surface; they therefore descend by a force which acts constantly and equally; consequently all the motions freely produced by gravity, are as above specified, by that proposition, &c.

SCHOLIUM.

70. Now it has been found, by numberless experiments, that

that gravity is a force of such a nature, that all bodies, whether light or heavy, fall perpendicularly through equal spaces in the same time, abstracting from the resistance of the air; as lead or gold and a feather, which in an exhausted receiver fall from the top to the bottom in the same time. It is also found that the velocities acquired by descending, are in the exact proportion of the times of descent: and further, that the spaces descended are proportional to the squares of the times, and therefore to the squares of the velocities. Hence then it follows, that the weights or gravities, of bodies near the surface of the earth, are proportional to the quantities of matter contained in them; and that the spaces, times, and velocities, generated by gravity, have the relations contained in the three general proportions before laid down. Further, as it is found, by accurate experiments, that a body in the latitude of London, falls nearly $16\frac{1}{12}$ feet in the first second of time, and consequently that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet by corol. 1, prop. 6; therefore, if g denote $16\frac{1}{12}$ feet, the space fallen through in one second of time, or $2g$ the velocity generated in that time; then, because the velocities are directly proportional to the times, and the spaces to the squares of the times; therefore it will be,

as $1'' : t'' :: 2g : 2gt = v$ the velocity,
and $1^2 : t^2 :: g : gt^2 = s$ the space.

So that, for the descents of gravity, we have these general equations, namely,

$$s = gt^2 = \frac{v^2}{4g} = \tfrac{1}{2}tv.$$

$$v = 2gt = \frac{2s}{t} = 2\sqrt{gs}.$$

$$t = \frac{v}{2g} = \frac{2s}{v} = \sqrt{\frac{s}{g}}.$$

$$g = \frac{v}{2t} = \frac{s}{t^2} = \frac{v^2}{4s}.$$

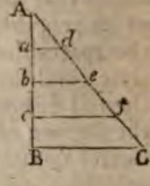
Hence, because the times are as the velocities, and the spaces as the squares of either, therefore,

if the times be as the numbs.	1, 2, 3, 4, 5, &c,
the velocities will also be as	1, 2, 3, 4, 5, &c,
and the spaces as their squares	1, 4, 9, 16, 25, &c,
and the space for each time as	1, 3, 5, 7, 9, &c,

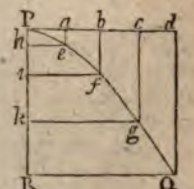
namely, as the series of the odd numbers, which are the differences of the squares denoting the whole spaces. So that if the first series of natural numbers be seconds of time,

namely,

namely, the times in seconds	1″,	2″,	3″,	4″,	&c
the velocities in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c
the spaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c
and the space for each second	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c



71. These relations, of the times, velocities, and spaces, may be aptly represented by certain lines and geometrical figures. Thus, if the line AB denote the time of any body's descent, and BC, at right angles to it, the velocity gained at the end of that time; by joining AC, and dividing the time AB into any number of parts at the points a, b, c; then shall ad, be, cf, parallel to BC, be the velocities at the points of time a, b, c, or at the ends of the times, Aa, Ab, AC; because these latter lines, by similar triangles, are proportional to the former ad, be, cf, and the times are proportional to the velocities. Also, the area of the triangle ABC will represent the space descended by the force of gravity in the time AB, in which it generates the velocity BC; because that area is equal to $\frac{1}{2}$AB × BC, and the space descended is $s = \frac{1}{2}tv$, or half the product of the time and the last velocity. And, for the same reason, the less triangles Aad, Abe, Acf, will represent the several spaces described in the corresponding times Aa, Ab, Ac, and velocities ad, be, cf; those triangles or spaces being also as the squares of their like sides Aa, Ab, Ac, which represent the times, or of ad, be, cf, which represent the velocities.



72. But as areas are rather unnatural representations of the spaces passed over by a body in motion, which are lines, the relations may better be represented by the abscisses and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate; and Pa, Pb, Pc, &c, parallel to RQ, represent the times from the beginning, or the velocities, then ae, bf, cg, &c, parallel to the axis PR, will represent the spaces described by a falling body in those times; for, in a parabola, the abscisses Ph, Pi, Pk, &c, or ae, bf, cg, &c, which are the spaces described, are as the squares of the ordinates he, if, kg, &c, or Pa, Pb, Pc, &c, which represent the times or velocities.

73. And because the laws for the destruction of motion, are

are the same as those for the generation of it, by equal forces, but acting in a contrary direction; therefore,

1*st*, A body thrown directly upward, with any velocity, will lose equal velocities in equal times.

2*d*, If a body be projected upward, with the velocity it acquired in any time by descending freely, it will lose all its velocity in an equal time, and will ascend just to the same height from which it fell, and will describe equal spaces in equal times, in rising and falling, but in an inverse order; and it will have equal velocities at any one and the same point of the line described, both in ascending and descending.

3*d*, If bodies be projected upward, with any velocities, the height ascended to, will be as the squares of those velocities, or as the squares of the times of ascending, till they lose all their velocities.

74. To illustrate now the rules for the natural descent of bodies by a few examples, let it be required,

1*st*, To find the space descended by a body in 7 seconds of time, and the velocity acquired.

Ans. $788\frac{1}{12}$ space; and $225\frac{1}{6}$ velocity.

2*d*, To find the time of generating a velocity of 100 feet per second, and the whole space descended.

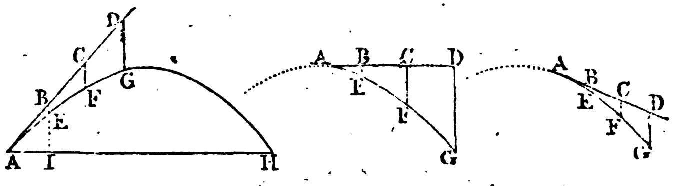
Ans. $3''\frac{21}{193}$ time; $155\frac{85}{193}$ space.

3*d*, To find the time of descending 400 feet, and the velocity at the end of that time.

Ans. $4''\frac{76}{77}$ time; and $160\frac{32}{77}$ velocity.

PROPOSITION XIX.

75. *If a Body be projected in Free Space, either Parallel to the Horizon, or in an Oblique Direction, by the Force of Gun-Powder, or any other Impulse; it will, by this Motion, in Conjunction with the Action of Gravity, describe the Curve Line of a Parabola.*



LET the body be projected from the point A, in the direction AD, with any uniform velocity: then, in any equal portions

portions of time, it would, by prop. 4, describe the equal spaces AB, BC, CD, &c, in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG, &c, in the direction of gravity, or perpendicular to the horizon, and equal to the spaces through which the body would descend by its gravity in the same time in which it would uniformly pass over the corresponding spaces AB, AC, AD, &c, by the projectile motion. Then, since by these two motions the body is carried over the space AB, in the same time as over the space BE, and the space AC in the same time as the space CF, and the space AD in the same time as the space DG, &c; therefore, by the composition of motions, at the end of those times, the body will be found respectively in the points E, F, G, &c; and consequently the real path of the projectile will be the curve line AEFG &c. But the spaces AB, AC, AD, &c, described by uniform motion, are as the times of description; and the spaces BE, CF, DG, &c, described in the same times by the accelerating force of gravity, are as the squares of the times; consequently the perpendicular descents are as the squares of the spaces in AD, that is BE, CF, DG, &c, are respectively proportional to AB^2, AC^2, AD^2, &c; which is the property of the parabola by theor. 8, Con. Sect. Therefore the path of the projectile is the parabolic line AEFG &c, to which AD is a tangent at the point A.

76. *Corol.* 1. The horizontal velocity of a projectile, is always the same constant quantity, in every point of the curve; because the horizontal motion is in a constant ratio to the motion in AD, which is the uniform projectile motion. And the projectile velocity is in proportion to the constant horizontal velocity, as radius to the cosine of the angle DAH, or angle of elevation or depression of the piece above or below the horizontal line AH.

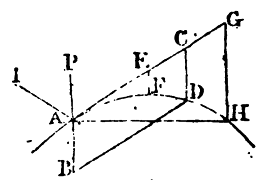
77. *Corol.* 2. The velocity of the projectile in the direction of the curve, or of its tangent at any point A, is as the secant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is constant, and AI is to AB, as radius to the secant of the angle A; therefore the motion at A, in AB, is everywhere as the secant of the angle A.

78 *Corol.* 3. The velocity in the direction DG of gravity, or perpendicular to the horizon, at any point G of the curve, is to the first uniform projectile velocity at A, or point of contact of a tangent, as 2GD is to AD. For, the times in AD and DG being equal, and the velocity acquired by freely descending

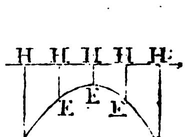
scending through DG, being such as would carry the body uniformly over twice DG in an equal time, and the spaces described with uniform motions being as the velocities, therefore the space AD is to the space 2DG, as the projectile velocity at A, to the perpendicular velocity at G.

PROPOSITION XX.

79. *The Velocity in the Direction of the Curve, at any Point of it, as* A, *is equal to that which is generated by Gravity in freely descending through a Space which is equal to One-Fourth of the Parameter of the Diameter of the Parabola at that Point.*



LET PA or AB be the height due to the velocity of the projectile at any point A, in the direction of the curve or tangent AC, or the velocity acquired by falling through that height; and complete the parallelogram ACDB. Then is CD = AB or AP, the height due to the velocity in the curve at A; and CD is also the height due to the perpendicular velocity at D, which must be equal to the former; but by the last corol. the velocity at A is to the perpendicular velocity at D, as AC to 2CD; and as these velocities are equal, therefore AC or BD is equal to 2CD, or 2AB; and hence AB or AP is equal to $\frac{1}{2}$BD, or $\frac{1}{4}$ of the parameter of the diameter AB, by corol. to theor. 13 of the Parabola.



80. *Corol.* 1. Hence, and from cor. 2, theor. 13 of the Parabola, it appears that, if from the directrix of the parabola which is the path of the projectile, several lines HE be drawn perpendicular to the directrix, or parallel to the axis; then the velocity of the projectile in the direction of the curve, at any point E, is always equal to the velocity acquired by a body falling freely through the perpendicular line HE.

81. *Corol.* 2. If a body, after falling through the height PA (last fig. but one), which is equal to AB, and when it arrives at A, have its course changed, by reflection from an elastic plane AI, or otherwise, into any direction AC, without altering the velocity; and if AC be taken = 2AP or 2AB, and

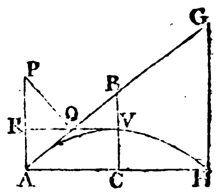
and the parallelogram be completed; then the body will describe the parabola passing through the point D.

82. *Corol.* 3. Because AC = 2AB or 2CD or 2AP, therefore $AC^2 = 2AP \times 2CD$ or AP . 4CD; and, because all the perpendiculars EF, CD, GH, are as AE^2, AC^2, AG^2; therefore also AP . 4EF = AE^2, and AP . 4GH = AG^2, &c; and, because the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore always

it is AP : AE :: AE : 4EF,
and AP : AC :: AC : 4CD,
and AP : AG :: AG : 4GH,
and so on.

PROPOSITION XXI.

83. *Having given the Direction, and the Impetus, or Altitude due to the First Velocity of a Projectile; to determine the Greatest Height to which it will rise, and the Random or Horizontal Range.*

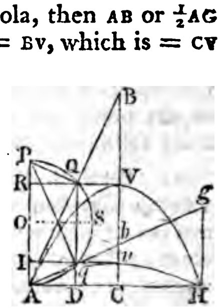


LET AP be the height due to the projectile velocity at A, AG the direction, and AH the horizon. On AG let fall the perpendicular PQ, and on AP the perpendicular QR; so shall AR be equal to the greatest altitude CV, and 4QR equal to the horizontal range AH. Or, having drawn PQ perp. to AG, take AG = 4AQ, and draw GH perp. to AH; then AH is the range.

For, by the last corollary,	AP : AG :: AG : 4GH;
and, by similar triangles,	AP : AG :: AQ : GH,
or - - -	AP : AG :: 4AQ : 4GH;

therefore AG = 4AQ; and, by similar triangles, AH = 4 Q.

Also, if V be the vertex of the parabola, then AB or $\frac{1}{2}$AG = 2AQ, or AQ = QB; consequently AR = BV, which is = CV by the property of the parabola.



84. *Corol.* 1. Because the angle Q is a right angle, which is the angle in a semicircle, therefore if, on AP as a diameter, a semicircle be described, it will pass through the point Q.

85. *Corol.* 2. If the horizontal range and the projectile velocity be given, the

the direction of the piece so as to hit the object H, will be thus easily found: Take AD = $\frac{1}{4}$AH, draw DQ perpendicular to AH, meeting the semicircle, described on the diameter AP, in Q and q; then AQ or Aq will be the direction of the piece. And hence it appears, that there are two directions AB, Ab, which, with the same projectile velocity, give the very same horizontal range AH. And these two directions make equal angles qAD, QAP with AH and AP, because the arc PQ = the arc Aq.

86. *Corol.* 3. Or, if the range AH, and direction AB, be given; to find the altitude and velocity or impetus. Take AD = $\frac{1}{4}$AH, and erect the perpendicular DQ, meeting AB in Q; so shall DQ be equal to the greatest altitude CV. Also, erect AP perpendicular to AH, and QP to AQ; so shall AP be the height due to the velocity.

87. *Corol.* 4. When the body is projected with the same velocity, but in different directions: the horizontal ranges AH will be as the sines of double the angles of elevation.— Or, which is the same, as the rectangle of the sine and cosine of elevation. For AD or RQ, which is $\frac{1}{4}$AH, is the sine of the arc AQ, which measures double the angle QAD of elevation.

And when the direction is the same, but the velocities different; the horizontal ranges are as the square of the velocities, or as the height AP, which is as the square of the velocity; for the sine AD or RQ or $\frac{1}{4}$AH is as the radius or as the diameter AP.

Therefore, when both are different, the ranges are in the compound ratio of the squares of the velocities, and the sines of double the angles of elevation.

88. *Corol.* 5. The greatest range is when the angle of elevation is 45°, or half a right angle; for the double of 45 is 90, which has the greatest sine. Or the radius OS, which is $\frac{1}{4}$ of the range, is the greatest sine.

And hence the greatest range, or that at an elevation of 45°, is just double the altitude AP which is due to the velocity, or equal to 4VC. Consequently, in that case, C is the focus of the parabola, and AH its parameter. Also, the ranges are equal, at angles equally above and below 45°.

89. *Corol.* 6. When the elevation is 15°, the double of which, or 30°, has its sine equal to half the radius; consequently then its range will be equal to AP, or half the greatest range at the elevation of 45°; that is, the range at 15°, is equal to the impetus or height due to the projectile velocity.

90. *Corol.* 7. The greatest altitude CV, being equal to AR, is as the versed sine of double the angle of elevation, and also as AP or the square of the velocity. Or as the square of the sine of elevation, and the square of the velocity; for the square of the sine is as the versed sine of the double angle.

91. *Corol.* 8. The time of flight of the projectile, which is equal to the time of a body falling freely through GH or 4CV, four times the altitude, is therefore as the square root of the altitude, or as the projectile velocity and sine of the elevation.

SCHOLIUM.

92. From the last proposition, and its corollaries, may be deduced the following set of theorems, for finding all the circumstances of projectiles on horizontal planes, having any two of them given. Thus, let s, c, t denote the sine, cosine, and tangent of elevation; S, v the sine and versed sine of the double elevation; R the horizontal range; T the time of flight; V the projectile velocity; H the greatest height of the projectile $g = 16\frac{1}{12}$ feet, and a the impetus, or the altitude due to the velocity V. Then,

$$\mathrm{R} = 2a\mathrm{S} = 4asc = \frac{\mathrm{SV}^2}{2g} = \frac{sc\mathrm{V}^2}{g} = \frac{gc\mathrm{T}^2}{s} = \frac{g\mathrm{T}^2}{t} = \frac{4\mathrm{H}}{t}.$$

$$\mathrm{V} = \sqrt{4ag} = \sqrt{\frac{2g\mathrm{R}}{\mathrm{S}}} = \sqrt{\frac{g\mathrm{R}}{sc}} = \frac{g\mathrm{T}}{s} = \frac{2}{s}\sqrt{g\mathrm{H}}.$$

$$\mathrm{T} = \frac{s\mathrm{V}}{g} = 2s\sqrt{\frac{a}{g}} = \sqrt{\frac{t\mathrm{R}}{g}} = \sqrt{\frac{s\mathrm{R}}{gc}} = 2\sqrt{\frac{\mathrm{H}}{g}}.$$

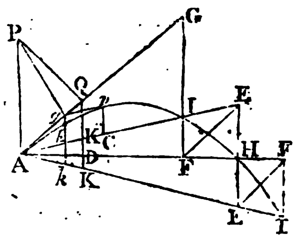
$$\mathrm{H} = as^2 = \tfrac{1}{2}av = \tfrac{1}{4}t\mathrm{R} = \frac{s\mathrm{R}}{4c} = \frac{s^2\mathrm{V}^2}{4g} = \frac{v\mathrm{V}^2}{8g} = \frac{g}{4}\mathrm{T}^2.$$

And from any of these, the angle of direction may be found. Also, in these theorems, g may, in many cases, be taken $= 16$, without the small fraction $\frac{1}{12}$, which will be near enough for common use.

PROPOSITION XXII.

93. *To determine the Range on an Oblique Plane; having given the Impetus or Velocity, and the Angle of Direction.*

LET AE be the oblique plane, at a given angle, either above or below the horizontal plane AH; AG the direction of



of the piece, and AP the altitude due to the projectile velocity at A.

By the last proposition, find the horizontal range AH to the given velocity and direction; draw HE perpendicular to AH, meeting the oblique plane in E; draw EF parallel to AG, and FI parallel to HE; so shall the projectile pass through I, and the range on the oblique plane will be AI. As is evident by theor. 15 of the Parabola, where it is proved, that if AH, AI be any two lines terminated at the curve, and IF, HE parallel to the axis; then is EF parallel to the tangent AG.

94. *Otherwise*, without the Horizontal Range.

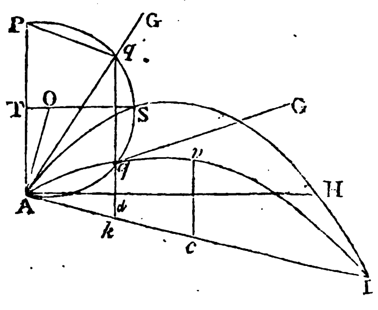
Draw PQ perp. to AG, and QD perp. to the horizontal plane AF, meeting the inclined plane in K; take AE = 4AK, draw EF parallel to AG, and PI parallel to AP or DQ; so shall AI be the range on the oblique plane. For AH = 4AD, therefore EH is parallel to FI, and so on, as above.

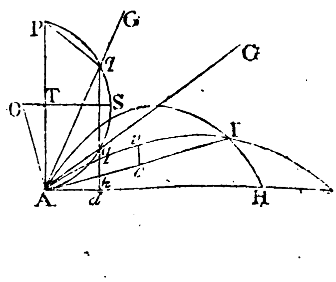
Otherwise.

95. Draw Pq making the angle APq = the angle GAI; then take AG = 4Aq, and draw GI perp. to AH. Or, draw qk perp. to AH, and take AI = 4A*k*. Also kq will be equal to *cv* the greatest height above the plane.

For, by cor. 2, prop. 20, AP : AG :: AG : 4GI;
and by sim. triangles, AP : AG :: Aq : GI,
or - - - AP : AG :: 4Aq : 4GI;
therefore AG = 4Aq; and by sim. triangles, AI = 4Ak.

Also, qk, or $\frac{1}{4}$GI, is = to *cv* by theor. 13 of the Parabola.





96. *Corol.* 1. If AO be drawn perp. to the plane AI, and

AP be bisected by the perpendicular STO; then with the centre O describing a circle through A and P, the same will also pass through q, because the angle GAI, formed by the tangent AI and AG, is equal to the angle APq, which will therefore stand on the same arc Aq.

97. *Corol.* 2. If there be given the range AI and the velocity, or the impetus, the direction will hence be easily found thus: Take $Ak = \frac{1}{4}AI$, draw kq perp. to AH, meeting the circle described with the radius AO in two points q and q; then Aq or Aq will be the direction of the piece. And hence it appears that there are two directions, which, with the same impetus, give the very same range AI. And these two directions make equal angles with AI and AP, because the arc Pq is equal the arc Aq. They also make equal angles with a line drawn from A through S, because the arc Sq is equal the arc Sq.

98. *Corol.* 3. Or, if there be given the range AI, and the direction Aq; to find the velocity or impetus. Take $Ak = \frac{1}{4}AI$, and erect kq perp. to AH, meeting the line of direction in q; then draw qP making the $\angle AqP = \angle Akq$; so shall AP be the impetus, or the altitude due to the projectile velocity.

99. *Corol.* 4. The range on an oblique plane, with a given elevation, is directly proportional to the rectangle of the cosine of the direction of the piece above the horizon, and the sine of the direction above the oblique plane, and reciprocally to the square of the cosine of the angle of the plane above or below the horizon.

For, put $s = \sin.\ \angle qAI$ or APq,

$c = \cos.\ \angle qAH$ or $\sin.\ PAq$,

$C = \cos.\ \angle IAH$ or $\sin.\ Akd$ or Akq or AqP.

Then, in the triangle APq, $C : s :: AP : Aq$;
and in the triangle Akq, $C : c :: Aq : Ak$;
theref. by composition, $C^2 : cs :: AP : AK = \frac{1}{4}AI$.

So that the oblique range $AI = \frac{cs}{C^2} \times 4AP$.

100. The range is the greatest when Ak is the greatest; that is, when kq touches the circle in the middle point S; and then the line of direction passes through S, and bisects the angle formed by the oblique plane and the vertex. Also, the ranges are equal at equal angles above and below this direction for the maximum.

101. *Corol.* 5. The greatest height *cv* or kq of the projectile

tile, above the plane, is equal to $\frac{s^2}{C^2} \times$ AP. And therefore it is as the impetus and square of the sine of direction above the plane directly, and square of the cosine of the plane's inclination reciprocally.

For - C (sin. AqP) : s (sin. APq) :: AP : Aq,
and C (sin. Akq) : s (sin. kAq) :: Aq : kq,
theref. by comp. $C^2 : s^2$ :: AP : kq.

102. *Corol.* 6. The time of flight in the curve AVI is $= \frac{2s}{C}\sqrt{\frac{AP}{g}}$, where $g = 16\frac{1}{12}$ feet. And therefore it is as the velocity and sine of direction above the plane directly, and cosine of the plane's inclination reciprocally. For the time of describing the curve, is equal to the time of falling freely through GI or 4kq or $\frac{4s^2}{C^2} \times$ AP. Therefore, the time being as the square root of the distance,

$\sqrt{g} : \frac{2s}{C}\sqrt{AP} :: 1'' : \frac{2s}{C}\sqrt{\frac{AP}{g}}$, the time of flight.

SCHOLIUM.

103. From the foregoing corollaries may be collected the following set of theorems, relating to projects made on any given inclined planes, either above or below the horizontal plane. In which the letters denote as before, namely,

c = cos. of direction above the horizon,
C = cos. of inclination of the plane,
s = sin. of direction above the plane,
R the range on the oblique plane,
T the time of flight,
V the projectile velocity,
H the greatest height above the plane,
a the impetus, or alt. due to the velocity V,
$g = 16\frac{1}{12}$ feet. Then,

$$R = \frac{cs}{C^2} \times 4a = \frac{cs}{C^2 g}V^2 = \frac{gc}{s}T^2 = \frac{4c}{s}H.$$

$$H = \frac{s^2}{C^2}a = \frac{s^2V^2}{4gC^2} = \frac{sR}{4c} = \frac{g}{4}T^2.$$

$$V = \sqrt{4ag} = C\sqrt{\frac{gR}{cs}} = \frac{gC}{s}T = \frac{2C}{s}\sqrt{gH}.$$

$$T = \frac{2s}{C}\sqrt{\frac{a}{g}} = \frac{sV}{gC} = \sqrt{\frac{sR}{gc}} = 2\sqrt{\frac{H}{g}}.$$

And from any of these, the angle of direction may be found.

FRAC-

PRACTICAL GUNNERY.

104. THE two foregoing propositions contain the whole theory of projectiles, with theorems for all the cases, regularly arranged for use, both for oblique and horizontal planes. But, before they can be applied to use in resolving the several cases in the practice of gunnery, it is necessary that some more data be laid down, as derived from good experiments made with balls or shells discharged from cannon or mortars, by gunpowder, under different circumstances. For, without such experiments and data, those theorems can be of very little use in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulsion of balls from guns, but more especially on account of the enormous resistance of the air to all projectiles made with any velocities that are considerable. As to the cases in which projectiles are made with small velocities, or such as do not exceed 200, or 300, or 400 feet per second of time, they may be resolved tolerably near the truth, especially for the larger shells, by the parabolic theory, laid down above. But, in cases of great projectile velocities, that theory is quite inadequate, without the aid of several data drawn from many and good experiments. For so great is the effect of the resistance of the air to projectiles of considerable velocity, that some of those which in the air range only between 2 and 3 miles at the most, would in vacuo range about ten times as far, or between 20 and 30 miles.

The effects of this resistance are also various, according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one size of ball or shell, will not serve for another size, though the velocity should be the same; neither will the experiments made with one velocity, serve for other velocities, though the ball be the same. And therefore it is plain that, to form proper rules for practical gunnery, we ought to have good experiments made with each size of mortar, and with every variety of charge, from the least to the greatest. And not only so, but these ought also to be repeated at many different angles of elevation, namely, for every single degree between 30° and 60° elevation, and at intervals of 5° above 60° and below 30°, from the vertical direction to point blank. By such a course of experiments it will be found, that the greatest range, instead of being constantly that at an elevation of 45°, as in the parabolic *theory*, will be at all intermediate degrees between 45 and 30, being

being more or less, both according to the velocity and the weight of the projectile; the smaller velocities and larger shells ranging farthest when projected almost at an elevation of 45°; while the greatest velocities, especially with the smaller shells, range farthest with an elevation of about 30°.

105. There have, at different times, been made certain small parts of such a course of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the sizes of mortars were used, and a variety of small charges of powder. But they were all at the elevation of 45°; consequently these are defective in the higher charges, and in all the other angles of elevation.

Other experiments were also carried on in the same place in the years 1784 and 1786, with various angles of elevation indeed, but with only one size of mortar, and only one charge of powder, and that but a small one too: so that all those nearly agree with the parabolic theory. Other experiments have also been carried on with the ballistic pendulum, at different times; from which have been obtained some of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the square root of the charge directly, and as the square root of the weight of ball reciprocally; and that, some rounds being fired with a medium length of one-pounder gun, at 15° and 45° elevation, and with 2, 4, 8, and 12 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here set down in the following Table.

Powder.	Elevation of gun.	Velocity of ball.	Range.	Time of flight.
oz.		feet.	feet.	
2	15°	860	4100	9″
4	15	1230	5100	12
8	15	1640	6000	$14\frac{1}{2}$
12	15	1680	6700	$15\frac{1}{2}$
2	45	860	5100	21

106. But as we are not yet provided with a sufficient number and variety of experiments, on which to establish true rules for practical gunnery, independent of the parabolic theory, we must at present content ourselves with the data of

some one certain experimented range and time of flight, at a given angle of elevation; and then by help of these, and the rules in the parabolic theory, determine the like circumstances for other elevations that are not greatly different from the former, assisted by the following practical rules.—

SOME PRACTICAL RULES IN GUNNERY.

I. *To find the Velocity of any Shot or Shell.*

RULE. Divide double the weight of the charge of powder by the weight of the shot, both in lbs. Extract the square root of the quotient. Multiply that root by 1600, and the product will be the velocity in feet, or the number of feet the shot passes over per second.

Or say—As the root of the weight of the shot, is to the root of double the weight of the powder, so is 1600 feet, to the velocity.

II. *Given the range at One Elevation; to find the Range at Another Elevation.*

RULE. As the sine of double the first elevation, is to its range; so is the sine of double another elevation, to its range.

III. *Given the Range for One Charge; to find the Range for Another Charge, or the Charge for Another Range.*

RULE. The ranges have the same proportion as the charges; that is, as one range is to its charge, so is any other range to its charge: the elevation of the piece being the same in both cases.

107. *Example* 1. If a ball of 1 lb. acquire a velocity of 1600 feet per second, when fired with 8 ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs. - -	196	90	48	16	8
Charge of powder in lbs. -	9	4	2	1	$\frac{1}{2}$
Ans. The velocities are -	485	477	462	566	566

108. *Exam.* 2. If a shell be found to range 1000 yards, when discharged at an elevation of 45°; how far will it range

range when the elevation is 30° 16′, the charge of powder being the same? Ans. 2612 feet, or 871 yards.

109. *Exam.* 3. The range of a shell, at 45° elevation, being found to be 3750 feet; at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder?

Ans. at 24° 16′, or at 65° 44′.

110. *Exam.* 4. With what impetus, velocity, and charge of powder, must a 13-inch shell be fired, at an elevation of 32° 12′, to strike an object at the distance of 3250 feet?

Ans. impetus 1802, veloc. 340, charge 4lb. $7\frac{1}{2}$oz.

111. *Exam.* 5. A shell being found to range 3500 feet, when discharged at an elevation of 25° 12′; how far then will it range at an elevation of 36° 15′ with the same charge of powder? Ans. 4332 feet.

112. *Exam.* 6. If, with a charge of 9lb. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases?

Ans. $6\frac{3}{4}$lb. of powder.

113. *Exam.* 7. What will be the time of flight for any given range, at the elevation of 45°?

Ans. the time in secs. is $\frac{1}{4}$ the sq. root of the range in feet.

114. *Exam.* 8. In what time will a shell range 3250 feet, at an elevation of 32°? Ans. $11\frac{1}{4}$ sec. nearly.

115. *Exam.* 9. How far will a shot range on a plane which ascends 8° 15′, and another which descends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′?

Ans. 4244 feet on the ascent,
and 6745 feet on the descent.

116. *Exam.* 10. How much powder will throw a 13-inch shell 4244 feet on an inclined plane, which ascends 8° 15′, the elevation of the mortar being 32° 30′?

Ans. 7·3765lb. or 7lb. 6oz.

117. *Exam.* 11. At what elevation must a 13-inch mortar be pointed, to range 6745 feet, on a plane which descends 8° 15′; the charge $7\frac{3}{8}$lb. of powder? Ans. 32° 28′.

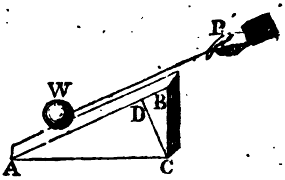
118. *Exam.* 12. In what time will a 13-inch shell strike a plane which rises 8° 30′, when elevated 45°, and discharged with an impetus of 2304 feet? Ans. $14\frac{2}{3}$ seconds.

THE DESCENT OF BODIES ON INCLINED PLANES AND CURVE SURFACES.—THE MOTION OF PENDULUMS.

PROPOSITION XXIII.

119. *If a weight* W *be Sustained on an Inclined Plane* AB, *by a Power* P, *acting in a Direction* WP, *Parallel to the Plane. Then*

The Weight of the Body, W	*The Length* AB,
The Sustaining Power P, *and*	*The Height* BC, *and*
The Pressure on the Plane, p,	*The Base* AC,
are respectively as	*of the Plane.*



FOR, draw CD perpendicular to the plane. Now here are three forces, keeping one another in equilibrio; namely, the weight, or force of gravity, acting perpendicular to AC, or parallel to BC; the power acting parallel to DB; and the pressure perpendicular to AB, or parallel to DC: but when three forces keep one another in equilibrio, they are proportional to the sides of the triangle CBD, made by lines in the direction of those forces, by prop. 8; therefore those forces are to one another as BC, BD, CD. But the two triangles ABC, CBD, are equiangular, and have their like sides proportional; therefore the three BC, BD, CD, are to one another respectively as the three AB, BC, AC; which therefore are as the three forces W, P, p.

120. *Corol.* 1. Hence the weight W, power P, and pressure p, are respectively as radius, sine, and cosine, of the plane's elevation BAC above the horizon.

For, since the sides of triangles are as the sines of their opposite angles, therefore the three AB, BC, AC,
are respectively as - - sin. C, sin. A, sin. B,
or as - - - - radius, sine, cosine,
of the angle A of elevation.

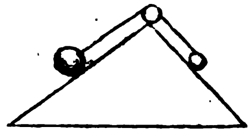
Or, the three forces are as AC, CD, AD; perpendicular to their directions.

121. *Corol.* 2. The power or relative weight that urges a body W down the inclined plane, is $= \frac{BC}{AB} \times W$; or the force

with

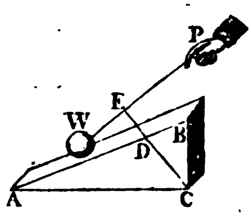
with which it descends, or endeavours to descend, is as the sine of the angle A of inclination.

122. *Corol.* 3. Hence, if there be two planes of the same height, and two bodies be laid on them which are proportional to the lengths of the planes; they will have an equal tendency to descend down the planes.



And consequently they will mutually sustain each other if they be connected by a string acting parallel to the planes.

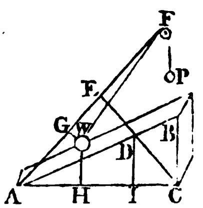
123. *Corol.* 4. In like manner, when the power P acts in any other direction whatever, WP; by drawing CDE perpendicular to the direction WP, the three forces in equilibrio, namely, the weight W, the power P, and the pressure on the plane, will still be respectively as AC, CD, AD, drawn perpendicular to the direction of those forces.



PROPOSITION XXIV.

124. *If a Weight* W *on an Inclined Plane* AB, *be in Equilibrio with another Weight* P *hanging freely; then if they be set a-moving, their Perpendicular Velocities, in that Place, will be Reciprocally as those Weights.*

LET the weight W descend a very small space, from W to A, along the plane, by which the string PFW will come into the position PFA. Draw WH perpendicular to the horizon AC, and WG perpendicular to AF: then WH will be the space perpendicularly descended by the weight W; and AG, or the difference between FA and FW, will be the space perpendicularly ascended by the weight P; and their perpendicular velocities are as those spa and AG passed over in those directions, in the same time. Draw CDE perpendicular to AF, and DI perpendicular to AC.



Then,

in the sim. figs. AGWH and AEDI,	AG : WH :: AE : DI;
and in the sim. tri. AEC, DIC,	AC : CD :: AE : DI;
but, by cor. 4, prop. 23,	AC : CD :: W : P;
therefore, by equality,	AG : WH :: W : P;

That

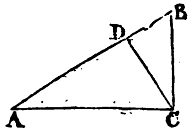
That is, their perpendicular spaces, or velocities, are reciprocally as their weights or masses.

125. *Corol.* 1. Hence it follows, that if any two bodies be in equilibrio on two inclined planes, and if they be set a-moving, their perpendicular velocity will be reciprocally as their weights. Because the perpendicular weight which sustains the one, would also sustain the other.

126. *Corol.* 2. And hence also, if two bodies sustain each other in equilibrio, on any planes, and they be put in motion; then each body multiplied by its perpendicular velocity, will give equal products.

PROPOSITION XXV.

127. *The Velocity acquired by a Body descending freely down an Inclined Plane* AB, *is to the Velocity acquired by a Body falling Perpendicularly, in the same Time; as the Height of the Plane* BC, *is to its Length* AB.



FOR the force of gravity, both perpendicularly and on the plane, is constant; and these two, by corol. 2, prop. 23, are to each other as AB to BC. But, by art. 28, the velocities generated by any constant forces, in the same time, are as those forces. Therefore the velocity down BA is to the velocity down BC, in the same time, as the force on BA to the force on BC: that is, as BC to BA.

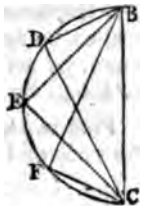
128. *Corol.* 1. Hence, as the motion down an inclined plane is produced by a constant force, it will be a motion uniformly accelerated; and therefore the laws before laid down for accelerated motions in general, hold good for motions on inclined planes; such, for instance, as the following: That the velocities are as the times of descending from rest; that the spaces descended are as the squares of the velocities, or squares of the times; and that if a body be thrown up an inclined plane, with the velocity it acquired in descending, it will lose all its motion, and ascend to the same height, in the same time, and will repass any point of the plane with the same velocity as it passed it in descending.

129. *Corol.* 2. Hence also, the space descended down an inclined plane, is to the space descended perpendicularly, in the same time, as the height of the plane CB, to its length AB, or as the sine of inclination to radius. For the spaces described

described by any forces, in the same time, are as the forces, or as the velocities.

130. *Corol.* 3. Consequently the velocities and spaces descended by bodies down different inclined planes, are as the sines of elevation of the planes.

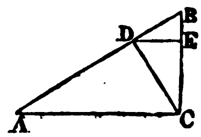
131. *Corol.* 4. If CD be drawn perpendicular to AB; then, while a body falls freely through the perpendicular space BC, another body will, in the same time, descend down the part of the plane BD. For by similar triangles, - - BC : BD :: BA : BC, that is, as the space descended, by corol. 2.



Or, in any right-angled triangle BDC, having its hypothenuse BC perpendicular to the horizon, a body will descend down any of its three sides BD, BC, DC, in the same time. And therefore, if on the diameter BC a circle be described, the time of descending down any chords BD, BE, BF, DC, EC, FC, &c, will be all equal, and each equal to the time of falling freely through the perpendicular diameter BC.

PROPOSITION XXVI.

132. *The Time of descending down the Inclined Plane* BA, *is to the Time of falling through the Height of the Plane* BC, *as the Length* BA *is to the Height* BC.



DRAW CD perpendicular to AB. Then the times of describing BD and BC are equal, by the last corol. Call that time t, and the time of describing BA call T.

Now, because the spaces described by constant forces, are as the squares of the times; therefore $t^2 : T^2 :: BD : BA$.

But the three BD, BC, BA, are in continual proportion; therefore $BD : BA :: BC^2 : BA^2$;
hence, by equality, $t^2 : T^2 :: BC^2 : BA^2$,
or - - - $t : T :: BC : BA$.

133. *Corol.* Hence the times of descending down different planes, of the same height, are to one another as the lengths of the planes.

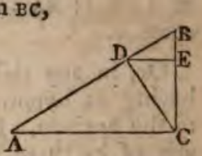
PROPOSITION

PROPOSITION XXVII.

134. *A Body acquires the Same Velocity in descending down any Inclined Plane* BA, *as by falling perpendicular through the Height of the Plane* BC.

FOR, the velocities generated by any constant forces, are in the compound ratio of the forces and times of acting. But if we put

F to denote the whole force of gravity in BC,
f the force on the plane AB,
t the time of describing BC, and
T the time of descending down AB;
then by art. 119, $F : f :: BA : BC$;
and by art. 132, $t : T :: BC : BA$;
theref. by comp. $Ft : fT :: 1 : 1$.



That is, the compound ratio of the forces and times, or the ratio of the velocities, is a ratio of equality.

135. *Corol.* 1. Hence the velocities acquired, by bodies descending down any planes, from the same height, to the same horizontal line, are equal.

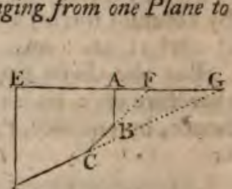
136. *Corol.* 2. If the velocities be equal, at any two equal altitudes, D, E; they will be equal at all other equal altitudes A, C.

137. *Corol.* 3. Hence also, the velocities acquired by descending down any planes, are as the square roots of the heights.

POPOSITION XXVIII.

138. *If a Body descend down any Number of Contiguous Planes,* AB, BC, CD; *it will at last acquire the Same Velocity, as a Body falling perpendicularly through the Same Height* ED, *supposing the Velocity not altered by changing from one Plane to another.*

PRODUCE the planes DC, CB, to meet the horizontal line EA produced in F and G. Then, by art. 135, the velocity at B is the same, whether the body descend through AB or FB. And therefore the velocity at C will be the same,



whether the body descend through ABC or through FC, which

which is also again, by art. 135, the same as by descending through GC. Consequently it will have the same velocity at D, by descending through the planes AB, BC, CD, as by descending through the plane GD; supposing no obstruction to the motion by the body impinging on the planes at B and C: and this again, is the same velocity as by descending through the same perpendicular height ED.

139. *Corol.* 1. If the lines ABCD, &c, be supposed indefinitely small, they will form a curve line, which will be the path of the body; from which it appears that a body acquires also the same velocity in descending along any curve, as in falling perpendicularly through the same height.

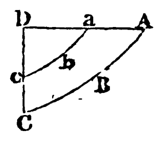
140. *Corol.* 2. Hence also, bodies acquire the same velocity by descending from the same height, whether they descend perpendicularly, or down any planes, or down any curve or curves. And if their velocities be equal, at any one height, they will be equal at all other equal heights. Therefore the velocity acquired by descending down any lines or curves, are as the square roots of the perpendicular heights.

141. *Corol.* 3. And a body, after its descent through any curve, will acquire a velocity which will carry it to the same height through an equal curve, or through any other curve, either by running up the smooth concave side, or by being retained in the curve by a string, and vibrating like a pendulum: Also, the velocities will be equal, at all equal altitudes; and the ascent and descent will be performed in the same time, if the curves be the same.

PROPOSITION XXIX.

142. *The Times in which Bodies descend through Similar Parts of Similar Curves,* ABC, abc, *placed alike, are as the Square Roots of their Lengths.*

THAT is, the time in AC is to the time in ac, as $\sqrt{AC}$ to $\sqrt{ac}$.



For, as the curves are similar, they may be considered as made up of an equal number of corresponding parts, which are every where, each to each, proportional to the whole. And as they are placed alike, the corresponding small similar parts will also be parallel to each other. But the time of describing each of these pairs of corresponding parallel parts, by art. 128, are as the square roots of their lengths,

lengths, which, by the supposition, are as $\sqrt{AC}$ to $\sqrt{ac}$, the roots of the whole curves. Therefore, the whole times are in the same ratio of $\sqrt{AC}$ to $\sqrt{ac}$.

143. *Corol.* 1. Because the axes DC, Dc, of similar curves, are as the lengths of the similar parts AC, ac; therefore the times of descent in the curves AC, ac, are as $\sqrt{DC}$ to $\sqrt{Dc}$, or the square roots of their axes.

144. *Corol.* 2. As it is the same thing, whether the bodies run down the smooth concave side of the curves, or be made to describe those curves by vibrating like a pendulum, the lengths being DC, Dc; therefore the times of the vibration of pendulums, in similar arcs of any curves, are as the square roots of the lengths of the pendulums.

SCHOLIUM.

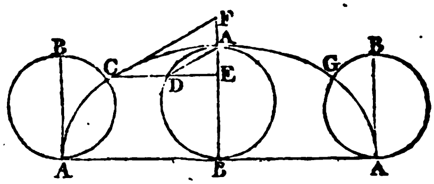
145. Having, in the last corollary, mentioned the pendulum, it may not be improper here to add some remarks concerning it.

A pendulum consists of a ball, or any other heavy body B, hung by a fine string or thread, moveable about a centre A, and describing the arc CBD; by which vibration the same motions happen to this heavy body, as would happen to any body descending by its gravity along the spherical superficies CBD, if that superficies were perfectly hard and smooth. If the pendulum be carried to the situation AC, and then let fall, the ball in descending will describe the arc CB; and in the point B it will have that velocity which is acquired by descending through CB, or by a body falling freely through EB. This velocity will be sufficient to cause the ball to ascend through an equal arc BD, to the same height D from whence it fell at C; having there lost all its motion, it will again begin to descend by its own gravity; and in the lowest point B it will acquire the same velocity as before; which will cause it to re-ascend to C: and thus, by ascending and descending, it will perform continual vibrations in the circumference CBD. And if the motions of pendulums met with no resistance from the air, and if there were no friction at the centre of motion A, the vibrations of pendulums would never cease. But from these obstructions, though small, it happens, that the velocity of the ball in the point B is a little diminished in every vibration; and consequently it does not return precisely to the same points C or D, but the arcs described continually

tinually become shorter and shorter, till at length they are insensible; unless the motion be assisted by a mechanical contrivance, as in clocks, called a maintaining power.

DEFINITION.

146. If the circumference of a circle be rolled on a right line, beginning at any point A, and continued till the same point A arrive at the line again, making just one revolution, and thereby measuring out a straight line ABA equal to the circumference of the circle, while the point A in the circumference traces out a curve line ACAGA; then this curve is called a cycloid; and some of its properties are contained in the following lemma.



LEMMA.

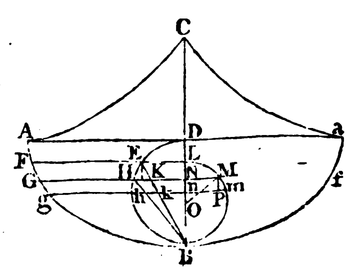
147. If the generating or revolving circle be placed in the middle of the cycloid, its diameter coinciding with the axis AB, and from any point there be drawn the tangent CF, the ordinate CDE perp. to the axis, and the chord of the circle AD: Then the chief properties are these:

The right line CD = the circular arc AD;
The cycloidal arc AC = double the chord AD;
The semi-cycloid ACA = double the diameter AB, and
The tangent CF is parallel to the chord AD.

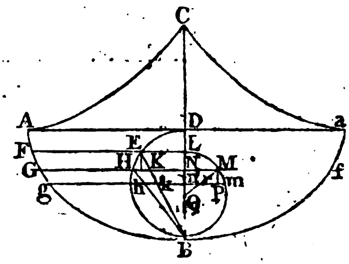
PROPOSITION XXX.

148. *When a Pendulum vibrates in a Cycloid; the Time of one Vibration, is to the Time in which a Body falls through Half the Length of the Pendulum, as the Circumference of a Circle is to its Diameter.*

LET ABa be the cycloid; DB its axis, or the diameter of the generating semicircle DEB; CB = 2DB the length of the pendulum, or radius of curvature at B. Let the ball descend from F, and, in vibrating, describe the arc FBf. Divide FB into innumerable small parts, one of which is Gg; draw FEL, GM, gm, perpendicular to



DB



DB. On LB describe the semicircle LMB, whose centre is O; draw Mp parallel to DB; also draw the chords BE, BH, EH, and the radius OM.

Now the triangles BEH, BHK, are equiangular; therefore BK : BH :: BH : BE, or $BH^2 = BK \cdot BE$, or $BH = \sqrt{(BK \cdot BE)}$.

And the equiangular triangles Mmp, MON, give Mp : Mm :: MN : MO. Also, by the nature of the cycloid, Hh is equal to Gg.

If another body descend down the chord EB, it will have the same velocity as the ball in the cycloid has at the same height. So that Kk and Gg are passed over with the same velocity, and consequently the time in passing them will be as their lengths Gg, Kk, or as Hh to Kk, or BH to BK by similar triangles, or $\sqrt{(BK \cdot BE)}$ to BK, or $\sqrt{BE}$ to $\sqrt{BK}$, or as $\sqrt{BL}$ to $\sqrt{BN}$ by similar triangles.

That is, the time in Gg : time in Kk :: $\sqrt{BL} : \sqrt{BN}$.

Again, the time of describing any space with a uniform motion, is directly as the space, and reciprocally as the velocity; also, the velocity in K or Kk, is to the velocity at B, as $\sqrt{EK}$ to $\sqrt{EB}$, or as $\sqrt{LN}$ to $\sqrt{LB}$; and the uniform velocity for EB is equal to half that at the point B, therefore the time in Kk : time in EB :: $\frac{Kk}{\sqrt{LN}} : \frac{EB}{\frac{1}{2}\sqrt{LB}} :: \frac{Nn}{\sqrt{LN}} : \frac{LB}{\frac{1}{2}\sqrt{LB}}$ (by sim. tri.) :: Nn or Mp : $2\sqrt{(BL \cdot LN.)}$

That is, the time in Kk : time in EB :: Mp : $2\sqrt{(BL \cdot LN)}$

But it was, time in Gg : time in Kk :: $\sqrt{BL} : \sqrt{BN}$; theref. by comp. time in Gg : time in EB :: Mp : $2\sqrt{(BN \cdot NL)}$ or 2NM.

But, by sim. tri. Mm : 2OM or BL :: Mp : 2NM.

Theref. time in Gg : time in EB :: Mm : BL.

Consequently the sum of all the times in all the Gg's, is to the time in EB, or the time in DB, which is the same thing, as the sum of all the Mm's, is to LB;

that is, the time in Fg : time in DB :: Lm : LB,
and the time in FB : time in DB :: LMB : LB,
or the time in FBf : time in DB :: 2LMB : LB.

That is, the time of one whole vibration,
is to the time of falling through half CB,
as the circumference of any circle,
is to its diameter.

149. *Corol.*

149. *Corol.* 1. Hence all the vibrations of a pendulum in a cycloid, whether great or small, are performed in the same time, which time is to the time of falling through the axis, or half the length of the pendulum, as 3·1416 to 1, the ratio of the circumference to its diameter; and hence that time is easily found thus. Put $p = 3·1416$, and l the length of the pendulum, also g the space fallen by a heavy body in 1″ of time.

then $\sqrt{g} : \sqrt{\frac{1}{2}l} :: 1'' : \sqrt{\frac{l}{2g}}$ the time of falling through $\frac{1}{2}l$,

theref. $1 : p :: \sqrt{\frac{l}{2g}} : p\sqrt{\frac{l}{2g}}$, which therefore is the time of one vibration of the pendulum.

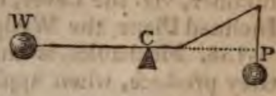
150. And if the pendulum vibrate in a small arc of a circle; because that small arc nearly coincides with the small cycloidal arc at the vertex B; therefore the time of vibration in the small arc of a circle, is nearly equal to the time of vibration in the cycloidal arc; consequently the time of vibration in a small circular arc, is equal to $p\sqrt{\frac{l}{2g}}$, where l is the radius of the circle.

151. So that, if one of these, g or l, be found by experiment, this theorem will give the other. Thus, if g, or the space fallen through by a heavy body in 1″ of time, be found, then this theorem will give the length of the second pendulum. Or, if the length of the second pendulum be observed by experiment, which is the easier way, this theorem will give g the descent of gravity in 1″. Now, in the latitude of London, the length of a pendulum which vibrates seconds, has been found to be $39\frac{1}{8}$ inches; and this being written for l in the theorem, it gives $p\sqrt{\frac{39\frac{1}{8}}{2g}} = 1''$: hence is found $g = \frac{1}{2}p^2 l = \frac{1}{2}p^2 \times 39\frac{1}{8} = 193·07$ inches $= 16\frac{1}{12}$ feet, for the descent of gravity in 1″; which it has also been found to be, very nearly, by many accurate experiments.

SCHOLIUM.

152. Hence is found the length of a pendulum that shall make any number of vibrations in a given time. Or, the number of vibrations that shall be made by a pendulum of a given length. Thus, suppose it were required to find the length of a half-seconds pendulum, or a quarter-seconds pendulum; that is, a pendulum to vibrate twice in a second, or 4 times in a second. Then, since the time of vibration is as the square root of the length,

166. A Fourth kind is sometimes added, called the Bended Lever. As a hammer drawing a nail.



167. In all these instruments the power may be represented by a weight, which is its most natural measure, acting downward: but having its direction changed, when necessary, by means of a fixed pulley.

PROPOSITION XXXI.

168. *When the Weight and Power keep the Lever in Equilibrio, they are to each other Reciprocally as the Distances of their Lines of Direction from the Prop. That is,* P : W :: CD : CE; *where* CD *and* CE *are perpendicular to* WO *and* AO, *the Directions of the two Weights, or the Weight and Power* W *and* A.

For, draw CF parallel to AO, and CB parallel to WO: Also, join CO, which will be the direction of the pressure on the prop C; for there cannot be an equilibrium unless the directions of the three forces all meet in, or tend to, the same point, as O. Then, because these three forces keep each other in equilibrio, they are proportional to the sides of the triangle CBO or CFO, drawn in the direction of those forces; therefore - - - - P : W :: CF : FO or CB.

But, because of the parallels, the two triangles CDF, CEB are equiangular, therefore - - - CD : CE :: CF : CB.
Hence, by equality, - - P : W :: CD : CE.

That is, each force is reciprocally proportional to the distance of its direction from the fulcrum.

And it will be found that this demonstration will serve for all the other kinds of levers, by drawing the lines as directed.

169. *Corol.* 1. When the angle A is = the angle W, then is CD : CE :: CW : CA :: P : W. Or when the two forces act perpendicularly on the lever, as two weights, &c; then, in case of an equilibrium, D coincides with W, and E with P; consequently then the above proportion becomes also P : W :: CW : CA, or the distances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to those forces.

170. *Corol.*

170. *Corol.* 2. If any force P be applied to a lever at A; its effect on the lever, to turn it about the centre of motion C, is as the length of the lever CA, and the sine of the angle of direction CAE. For the perp. CE is as CA × s. ∠A.

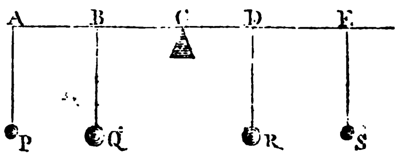
171. *Corol.* 3. Because the product of the extremes is equal to the product of the means, therefore the product of the power by the distance of its direction, is equal to the product of the weight by the distance of its direction.

That is, P × CE = W × CD.

172. *Corol.* 4. If the lever, with the weight and power fixed to it, be made to move about the centre C; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion to each other. For the weight and power will describe circles whose radii are the distances CD, CE; and since the circumferences or spaces described, are as the radii, and also as the velocities, therefore the velocities are as the radii CD, CE; and the momenta, which are as the masses and velocities, are as the masses and radii; that is, as P × CE and W × CD; which are equal by cor. 3.

173. *Corol.* 5. In a straight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of these three, the power, weight, and pressure on the prop, any one is as the distance of the other two.

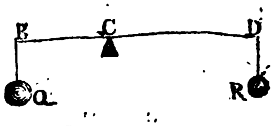
174. *Corol.* 6. If several weights P, Q, R, S, act on a straight lever, and keep it in equilibrio; then the sum of the products on one side of the prop, will be equal to the sum on the other side, made by multiplying each weight by its distance; namely,



P × AC + Q × BC = R × DC + S × EC.

For, the effect of each weight to turn the lever, is as the weight multiplied by its distance; and in the case of an equilibrium, the sums of the effects, or of the products on both sides, are equal.

175. *Corol.* 7. Because, when two weights Q and R are in equilibrio, Q : R :: CD : CB;

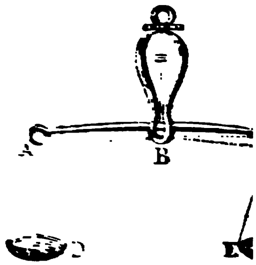


therefore, by composition, Q + R : Q :: BD : CD, and, Q + R : R :: BD : CB.

That is, the sum of the weights is to either of th
the sum of their distances is to the distance of the oth

SCHOLIUM.

176. On the foregoing principles depends the nature of scales and beams, for weighing all sorts of goods. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties:



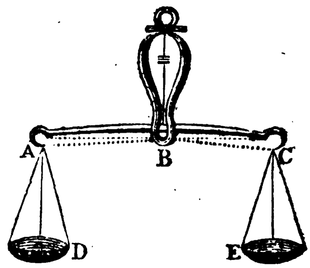
1st, That the points of suspension of the scales an
centre of motion of the beam, A, B, C, should be in a str
line: 2d, That the arms AB, BC, be of an equal length
That the centre of gravity be in the centre of moti
or a little below it: 4th, That they be in equilibrio
empty: 5th, That there be as little friction as possible a
centre B. A defect in any of these properties, makes
scales either imperfect or false. But it often happens
the one side of the beam is made shorter than the other
the defect covered by making that side the heavier
which means the scales hang in equilibrio when empty;
when they are charged with any weights so as to be st
equilibrio, those weights are not equal; but the deceit
be detected by changing the weights to the contrary
for then the equilibrium will be immediately destroyed.

177. To find the true weight of any body by such a
balance:—First weigh the body in one scale, and afterw
weigh it in the other; then the mean proportional betw
these two weights, will be the true weight required. F
any body b weigh w pounds or ounces in the scale D,
only v pounds or ounces in the scale E: then we have
two equations, namely, $AB . b = BC . w$,
and $BC . b = AB . v$;
the product of the two is $AB . BC . b^2 = AB . BC . wv$
hence then - - - - $b^2 = wv$,
and - - - - $b = \sqrt{wv}$,
[illegible] proportional, which is the true weight of the
[illegible] Statera, or Steelyard, is also
[illegible] arms, so contrived
[illegible] a great [illegible]

That is, the sum of the weights is to either of them, as the sum of their distances is to the distance of the other.

SCHOLIUM.

176. On the foregoing principles depends the nature of scales and beams, for weighing all sorts of goods. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties:



1*st*, That the points of suspension of the scales and the centre of motion of the beam, A, B, C, should be in a straight line: 2*d*, That the arms AB, BC, be of an equal length: 3*d*, That the centre of gravity be in the centre of motion B, or a little below it: 4*th*, That they be in equilibrio when empty: 5*th*, That there be as little friction as possible at the centre B. A defect in any of these properties, makes the scales either imperfect or false. But it often happens that the one side of the beam is made shorter than the other, and the defect covered by making that scale the heavier, by which means the scales hang in equilibrio when empty; but when they are charged with any weights, so as to be still in equilibrio, those weights are not equal; but the deceit will be detected by changing the weights to the contrary sides, for then the equilibrium will be immediately destroyed.

177. To find the true weight of any body by such a false balance:—First weigh the body in one scale, and afterwards weigh it in the other; then the mean proportional between these two weights, will be the true weight required. For, if any body b weigh W pounds or ounces in the scale D, and only w pounds or ounces in the scale E: then we have these two equations, namely, $AB \cdot b = BC \cdot W$.

and $BC \cdot b = AB \cdot w$;

the product of the two is $AB \cdot BC \cdot b^2 = AB \cdot BC \cdot Ww$;

hence then - - - $b^2 = Ww$,

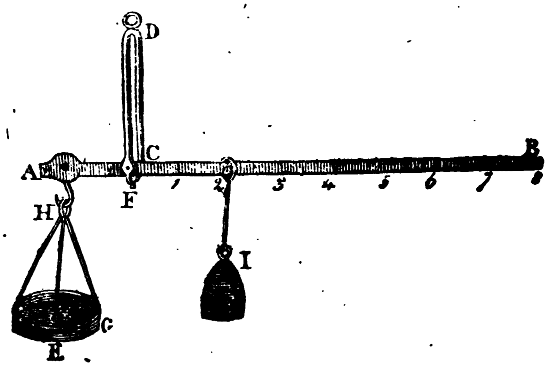
and - - - $b = \sqrt{Ww}$,

the mean proportional, which is the true weight of the body b.

178. The Roman Statera, or Steelyard, is also a lever, but of unequal brachia or arms, so contrived, that one weight only may serve to weigh a great many, by sliding it back-

ward

ward and forward, to different distances, on the longer arm of the lever; and it is thus constructed:



Let AB be the steelyard, and C its centre of motion, whence the divisions must commence if the two arms just balance each other: if not, slide the constant moveable weight I along from B towards C, till it just balance the other end without a weight, and there make a notch in the beam, marking it with a cipher 0. Then hang on at A a weight W equal to I, and slide I back towards B till they balance each other; there notch the beam, and mark it with 1. Then make the weight W double of I, and sliding I back to balance it, there mark it with 2. Do the same at 3, 4, 5, &c, by making W equal to 3, 4, 5, &c, times I; and the beam is finished. Then, to find the weight of any body *b* by the steelyard; take off the weight W, and hang on the body *b* at A; then slide the weight I backward and forward till it just balance the body *b*, which suppose to be at the number 5; then is *b* equal to 5 times the weight of I. So, if I be one pound, then *b* is 5 pounds; but if I be 2 pounds, then *b* is 10 pounds; and so on.

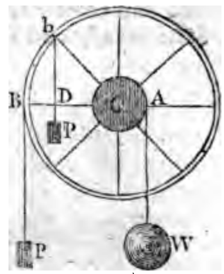
OF THE WHEEL AND AXLE.

PROPOSITION XXXII.

179. *In the Wheel-and-Axle; the Weight and Power will be in Equilibrio, when the Power* P *is to the Weight* W, *Reciprocally as the Radii of the Circles where they act; that is, as the Radius of the Axle* CA, *where the Weight hangs, to the Radius of the Wheel* CB, *where the Power acts. That is,* P : W :: CA : CB.

HERE the cord, by which the power P acts, goes about the

the circumference of the wheel, while that of the weight W goes round its axle, or another smaller wheel, attached to the larger, and having the same axis or centre C. So that BA is a lever moveable about the point C, the power P acting always at the distance BC, and the weight W at the distance CA; therefore P : W :: CA : CB.

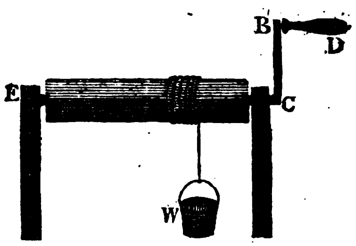


180. *Corol.* 1. If the wheel be put in motion; then, the spaces moved being as the circumferences, or as the radii, the velocity of W will be to the velocity of P, as CA to CB; that is, the weight is moved as much slower, as it is heavier than the power; so that what is gained in power, is lost in time. And this is the universal property of all machines and engines.

181. *Corol.* 2. If the power do not act at right angles to the radius cb, but obliquely; draw CD perpendicular to the direction of the power; then, by the nature of the lever, P : W :: CA : CD.

SCHOLIUM.

182. To this power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or spindle CE, by the handle CBD; the power applied at B is to the weight W on the roller, as the radius of the roller is to the radius CB of the handle.



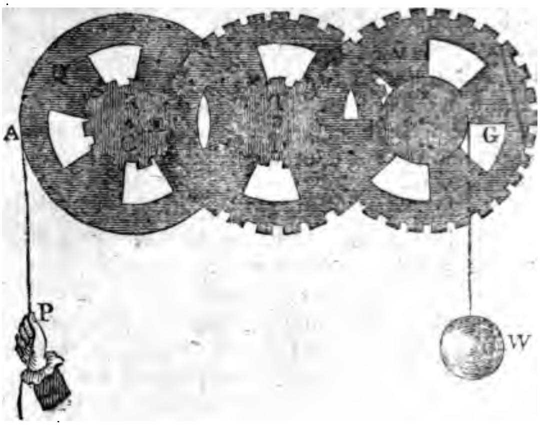
183. And the same for all cranes, capstans, windlasses, and such like; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts; so that they are always in the reciprocal ratio of their velocities. And to the same principle may be referred the gimblet and augur for boring holes.

184. But all this, however, is on supposition that the ropes or cords, sustaining the weights, are of no sensible thickness. For, if the thickness be considerable, or if there be several folds of them, over one another, on the roller or barrel; then we must measure to the middle of the outermost rope, for the

the radius of the roller; or, to the radius of the roller we must add half the thickness of the cord, when there is but one fold.

185. The wheel-and-axle has a great advantage over the simple lever, in point of convenience. For a weight can be raised but a little way by the lever; whereas, by the continual turning of the wheel and roller, the weight may be raised to any height, or from any depth.

186. By increasing the number of wheels too, the power may be multiplied to any extent, making always the less wheels to turn greater ones, as far as we please: and this is commonly called Tooth and Pinion Work, the teeth of one circumference working in the rounds or pinions of another, to turn the wheel. And then, in case of an equilibrium, the power is to the weight, as the continual product of the radii of all the axles, to that of all the wheels. So, if the power P



turn the wheel Q, and this turn the small wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V; and this turn the axle X, which raises the weight W; then P : W :: CB . DE . FG : AC . BD . EF. And in the same proportion is the velocity of W slower than that of P. Thus, if each wheel be to its axle, as 10 to 1; then P : W :: 1^3 : 10^3 or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds; but then, when put in motion, the power will move 1000 times faster than the weight.

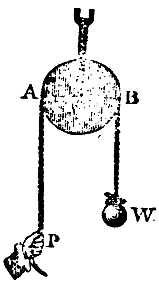
OF

Of the PULLEY.

187. A Pulley is a small wheel, commonly made of wood or brass, which turns about an iron axis passing through the centre, and fixed in a block, by means of a cord passed round its circumference, which serves to draw up any weight. The pulley is either single, or combined together, to increase the power. It is also either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power.

PROPOSITION XXXIII.

188. *If a Power sustain a Weight by means of a Fixed Pulley; the Power and Weight are Equal.*



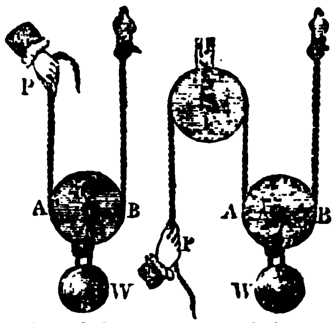
For through the centre c of the pulley draw the horizontal diameter AB: then will AB represent a lever of the first kind, its prop being the fixed centre c; from which the points A and B, where the power and weight act, being equally distant, the power P is consequently equal to the weight W.

189. *Corol.* Hence, if the pulley be put in motion, the power P will descend as fast as the weight W ascends. So that the power is not increased by the use of the fixed pulley, even though the rope go over several of them. It is, however, of great service in the raising of weights, both by changing the direction of the force, for the convenience of acting, and by enabling a person to raise a weight to any height without moving from his place, and also by permitting a great many persons at once to exert their force on the rope at P, which they could not do to the weight itself; as is evident in raising the hammer or weight of a pile-driver, as well as on many other occasions.

PROPOSITION XXXIV.

190. *If a Power sustain a Weight by means of One Moveable Pulley; the Power is but Half the Weight.*

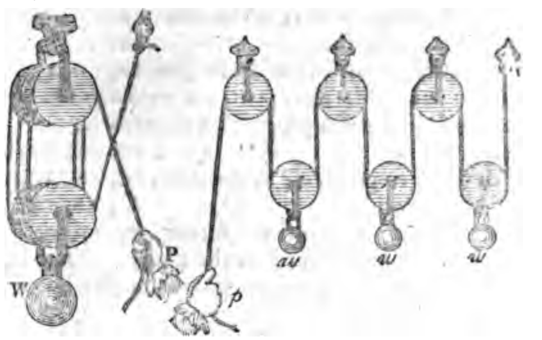
For, here AB may be considered as a lever of the second kind,



kind, the power acting at A, the weight at C, and the prop or fixed point at B; and because P : W :: CB : AB, and CB = $\frac{1}{2}$AB, therefore P = $\frac{1}{2}$W, or W = 2P.

191. *Corol.* 1. Hence it is evident, that, when the pulley is put in motion, the velocity of the power will be double the velocity of the weight, as the point P moves twice as fast as the point C and weight W rises. It is also evident, that the fixed pulley F makes no difference in the power P, but is only used to change the direction of it, from upwards to downwards.

192. *Corol.* 2. Hence we may estimate the effect of a combination of any number of fixed and moveable pulleys; by which we shall find that every cord going over a moveable pulley always adds 2 to the powers; since each moveable pulley's rope bears an equal share of the weight; while each rope that is fixed to a pulley, only increases the power by unity.



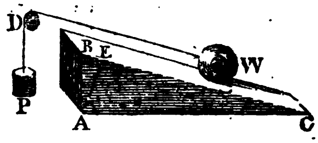
Here P = $\frac{1}{6}$W.   Here $p = \frac{1}{2}w = \frac{w+w+w}{6}$

OF THE INCLINED PLANE.

193. THE INCLINED PLANE, is a plane inclined to the horizon, or making an angle with it. It is often reckoned one of the simple mechanic powers; and the double inclined plane makes the wedge. It is employed to advantage in raising heavy bodies in certain situations, diminishing their weights by laying them on the inclined planes.

PROPOSITION XXXV.

194. *The Power gained by the Inclined Plane, is in Proportion as the Length of the Plane is to its Height. That is, when a Weight* W *is sustained on an Inclined Plane* BC, *by a Power* P *acting in the Direction* DW, *parallel to the Plane; then the Weight* W, *is in proportion to the Power* P, *as the Length of the Plane is to its Height; that is,* W : P :: BC : AB.



For, draw AE perp. to the plane BC, or to DW. Then we are to consider that the body W is sustained by three forces, viz. 1st, its own weight or the force of gravity, acting perp. to AC, or parallel to BA; 2d, by the power P, acting in the direction WD, parallel to BC, or BE; and 3dly, by the re-action of the plane, perp. to its face, or parallel to the line EA. But when a body is kept in equilibrio by the action of three forces, it has been proved, that the intensities of these forces are proportional to the sides of the triangle ABE, made by lines drawn in the directions of their actions; therefore those forces are to one another as the three lines - - - - AB, BE, AE; that is,
the weight of the body W is as the line AB,
the power P is as the line - - BE,
and the pressure on the plane as the line AE.
But the two triangles ABE, ABC are equiangular, and have therefore their like sides proportional; that is,
the three lines - - - - AB, BE, AE,
are to each other respectively as the three BC, AB, AC,
or also as the three - - - BC, AE, CE,
which therefore are as the three forces W, P, p,
where p denotes the pressure on the plane. That is, W : P :: BC : AB, or the weight is to the power, as the length of the plane is to its height.

See more on the Inclined Plane, at p. 164, &c.

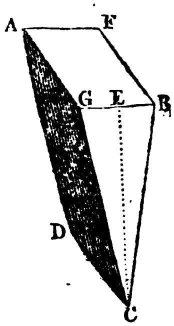
195. *Scholium*. The Inclined plane comes into use in some situations in which the other mechanical powers cannot be conveniently applied, or in combination with them. As, in sliding heavy weights either up or down a plank or other plane laid sloping: or letting large casks down into a cellar, or drawing them out of it. Also, in removing earth from a lower situation to a higher by means of wheel-barrows, or otherwise, as in making fortifications, &c; inclined planes, made of boards, laid aslope, serve for the barrows to run upon.

Of

Of all the various directions of drawing bodies up an inclined plane, or sustaining them on it, the most favourable is where it is parallel to the plane BC, and passing through the centre of the weight; a direction which is easily given to it, by fixing a pulley at D, so that a cord passing over it, and fixed to the weight, may act or draw parallel to the plane. In every other position, it would require a greater power to support the body on the plane, or to draw it up. For if one end of the line be fixed at W, and the other end inclined down towards B, below the direction WD, the body would be drawn down against the plane, and the power must be increased in proportion to the greater difficulty of the traction. And, on the other hand, if the line were carried above the direction of the plane, the power must be also increased; but here only in proportion as it endeavours to lift the body off the plane.

If the length BC of the plane be equal to any number of times its perp. height AB, as suppose 3 times; then a power P of 1 pound, hanging freely, will balance a weight W of 3 pounds, laid on the plane; and a power P of 2 pounds, will balance a weight W of 6 pounds; and so on, always 3 times as much. But then if they be set a-moving, the perp. descent of the power P, will be equal to 3 times as much as the perp. ascent of the weight W. For, though the weight W ascends up the direction of the oblique plane, BC, just as fast as the power P descends perpendicularly, yet the weight rises only the perp. height AB, while it ascends up the whole length of the plane BC, which is 3 times as much; that is, for every foot of the perp. rise of the weight, it ascends 3 feet up in the direction of the plane, and the power P descends just as much, or 3 feet.

OF THE WEDGE.



196. THE WEDGE is a piece of wood or metal, in form of half a rectangular prism. AF or BG is the breadth of its back; CE its height; GC, BC its sides; and its end GBC is composed of two equal inclined planes GCE, BCE.

PROPOSITION

PROPOSITION XXXVI.

197. *When a Wedge is in Equilibrio; the Power acting against the Back, is to the Force acting Perpendicularly against either Side, as the Breadth of the Back* AB, *is to the Length of the Side* AC *or* BC.

FOR, any three forces, which sustain one another in equilibrio, are as the corresponding sides of a triangle drawn perpendicular to the directions in which they act. But AB is perp. to the force acting on the back, to urge the wedge forward; and the sides AC, BC are perp. to the forces acting on them; therefore the three forces are as AB, AC, BC.

198. *Corol.* The force on the back,
Its effect in direct. perp. to AC,
And its effect parallel to AB;
are as the three lines

$\left\{\begin{array}{l}\text{AB,}\\ \text{AC,}\\ \text{DC,}\end{array}\right.$ which are per. to them,

And therefore the thinner a wedge is, the greater is its effect, in splitting any body, or in overcoming any resistance against the sides of the wedge.

SCHOLIUM.

199. But it must be observed, that the resistance, or the forces above-mentioned, respect one side of the wedge only. For if those against both sides be taken in, then, in the foregoing proportions, we must take only half the back AD, or else we must take double the line AC or DC.

In the wedge, the friction against the sides is very great, at least equal to the force to be overcome, because the wedge retains any position to which it is driven; and therefore the resistance is doubled by the friction. But then the wedge has a great advantage over all the other powers, arising from the force of percussion or blow with which the back is struck, which is a force incomparably greater than any dead weight or pressure, such as is employed in other machines. And accordingly we find it produces effects vastly superior to those of any other power; such as the splitting and raising the largest and hardest rocks, the raising and lifting the largest ship, by driving a wedge below it, which a man can do by the blow of a mallet: and thus it appears that the small blow of a hammer, on the back of a wedge, is incomparably greater than any mere pressure, and will overcome it.

Of

OF THE SCREW.

200. THE SCREW is one of the six mechanical powers, chiefly used in pressing or squeezing bodies close, though sometimes also in raising weights.

The screw is a spiral thread or groove cut round a cylinder, and everywhere making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder, is to the distance between two threads of the screw: as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.

PROPOSITION XXXVII.

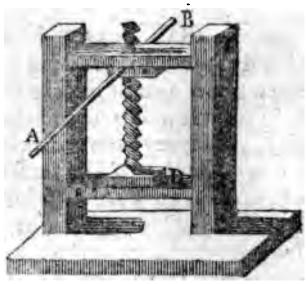
201. *The Force of a Power applied to turn a Screw round, is to the Force with which it presses upward or downward, setting aside the Friction, as the Distance between two Threads, is to the Circumference where the Power is applied.*

THE screw being an inclined plane, or half wedge, whose height is the distance between two threads, and its base the circumference of the screw; and the force in the horizontal direction, being to that in the vertical one, as the lines perpendicular to them, namely, as the height of the plane, or distance of the two threads, is to the base of the plane, or circumference of the screw; therefore the power is to the pressure, as the distance of two threads is to that circumference. But, by means of a handle or lever, the gain in power is increased in the proportion of the radius of the screw to the radius of the power, or length of the handle, or as their circumferences. Therefore, finally, the power is to the pressure, as the distance of the threads, is to the circumference described by the power.

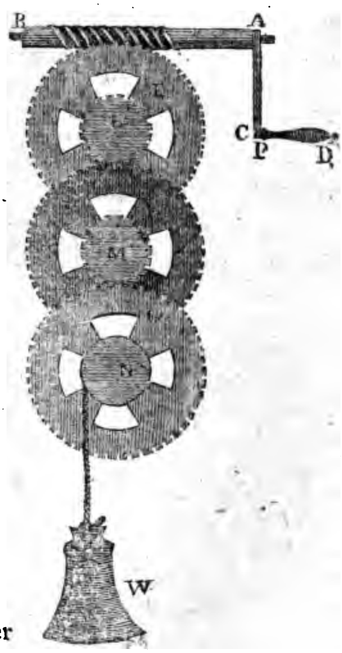
202. *Corol.* When the screw is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio; or that each of them is reciprocally proportional to its velocity.

SCHO

SCHOLIUM.



203. Hence we can easily compute the force of any machine turned by a screw. Let the annexed figure represent a press driven by a screw, whose threads are each a quarter of an inch asunder; and let the screw be turned by a handle of 4 feet long, from A to B; then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be required to determine with what force the screw will press on the board at D, when the man turns the handle at A and B, with his whole force. Then the diameter AB of the power being 4 feet, or 48 inches, its circumference is $48 \times 3{\cdot}1416$ or $150\frac{4}{5}$ nearly; and the distance of the threads being $\frac{1}{4}$ of an inch; therefore the power is to the pressure, as 1 to $603\frac{1}{5}$; but the power is equal to 150lb; theref. as $1 : 603\frac{1}{5} :: 150 : 90480$; and consequently the pressure at D is equal to a weight of 90480 pounds, independent of friction.



204. Again, if the endless screw AB be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turns a toothed wheel E, whose pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raise, working at the handle C; supposing the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches; the teeth and pinions being all of a size.

Her

Here $20 \times 3{\cdot}1416 \times 2 = 125{\cdot}664$, is the circumference of the power.

And $125{\cdot}664$ to $\frac{1}{2}$, or $251{\cdot}328$ to 1, is the force of the screw alone.

Also, 18 to 2, or 9 to 1, being the proportion of the wheels to the pinions; and as there are three of them, therefore 9^3 to 1^3, or 729 to 1, is the power gained by the wheels.

Consequently $251{\cdot}328 \times 729$ to 1, or $183218\frac{1}{5}$ to 1 nearly, is the ratio of the power to the weight, arising from the advantage both of the screw and the wheels.

But the power is 150lb; therefore $150 \times 183218\frac{1}{5}$, or 27482716 pounds, is the weight the man can sustain, which is equal to 12269 tons weight.

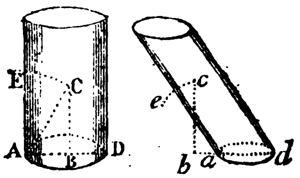
But the power has to overcome, not only the weight, but also the friction of the screw, which is very great, in some cases equal to the weight itself, since it is sometimes sufficient to sustain the weight, when the power is taken off.

ON THE CENTRE OF GRAVITY.

205. THE CENTRE of GRAVITY of a body, is a certain point within it, on which the body being freely suspended, it will rest in any position; and it will always descend to the lowest place to which it can get, in other positions.

PROPOSITION XXXVIII.

206. *If a Perpendicular to the Horizon, from the Centre of Gravity of any Body, fall Within the Base of the Body, it will rest in that Position; but if the Perpendicular fall Without the Base, the Body will not rest in that Position, but will tumble down.*



FOR, if CB, be the perp. from the centre of gravity C, within the base: then the body cannot fall over towards A; because, in turning on the point A, the centre of gravity C would describe an arc which would rise from C to E; contrary to the nature of that centre, which only rests when in the lowest place. For the same reason, the body will not fall towards D. And therefore it will stand in that position.

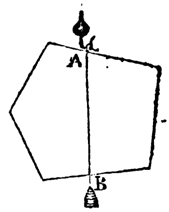
But

But if the perpendicular fall without the base, as cb; then the body will tumble over on that side: because, in turning on the point a, the centre c descends by describing the descending arc ce.

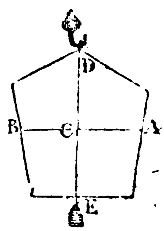
207. *Corol.* 1. If a perpendicular, drawn from the centre of gravity, fall just on the extremity of the base; the body may stand; but any the least force will cause it to fall that way. And the nearer the perpendicular is to any side, or the narrower the base is, the easier it will be made to fall, or be pushed over that way; because the centre of gravity has the less height to rise: which is the reason that a globe is made to roll on a smooth plane by any the least force. But the nearer the perpendicular is to the middle of the base, or the broader the base is, the firmer the body stands.

208. *Corol.* 2. Hence if the centre of gravity of a body be supported, the whole body is supported. And the place of the centre of gravity must be accounted the place of the body; for into that point the whole matter of the body may be supposed to be collected, and therefore all the force also with which it endeavours to descend.

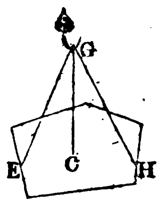
209. *Corol.* 3. From the property which the centre of gravity has, of always descending to the lowest point, is derived an easy mechanical method of finding that centre.



Thus, if the body be hung up by any point A, and a plumb line AB be hung by the same point, it will pass through the centre of gravity; because that centre is not in the lowest point till it fall in the plumb line. Mark the line AB on it. Then hang the body up by any other point D, with a plumb line DE, which will also pass through the centre of gravity, for the same reason as before; and therefore that centre must be at C where the two plumb lines cross each other.



210. Or, if the body be suspended by two or more cords GF, GH, &c, then a plumb line from the point G will cut the body in its centre of gravity C.



211. Like-

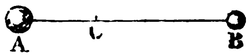
211. Likewise, because a body rests when its centre of gravity is supported, but not else; we hence derive another easy method of finding that centre mechanically. For, if the body be laid on the edge of a prism, or over one side of a table, and moved backward and forward till it rest, or balance itself; then is the centre of gravity just over the line of the edge. And if the body be then shifted into another position, and balanced on the edge again, this line will also pass by the centre of gravity; and consequently the intersection of the two will give the centre itself.

PROPOSITION XXXIX.

212. *The common Centre of Gravity* C *of any two Bodies* A, B, *divides the Line joining their Centres, into two Parts, which are Reciprocally as the Bodies.*

That is, AC : BC :: B : A.

FOR, if the centre of gravity C be supported, the two bodies A and B will be supported, and will rest in equilibrio. But, by the nature of the lever, when



two bodies are in equilibrio about a fixed point C, they are reciprocally as their distances from that point; therefore A : B :: CB : CA.

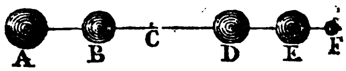
213. *Corol.* 1. Hence AB : AC :: A + B : B; or, the whole distance between the two bodies, is to the distance of either of them from the common centre, as the sum of the bodies is to the other body.

214. *Corol.* 2. Hence also, CA . A = CB . B; or the two products are equal, which are made by multiplying each body by its distance from the centre of gravity.

215. *Corol.* 3. As the centre C is pressed with a force equal to both the weights A and B, while the points A and B are each pressed with the respective weights A and B. Therefore, if the two bodies be both united in their common centre C, and only the ends A and B of the line AB be supported, each will still bear, or be pressed by the same weights A and B as before. So that, if a weight of 100lb. be laid on a bar at C, supported by two men at A and B, distant from C, the one 4 feet, and the other 6 feet; then the nearer will bear the weight of 60lb, and the farther only 40lb. weight.

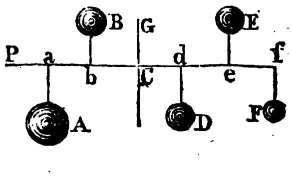
216. Corol

216. *Corol.* 4. Since the effect of any body to turn a lever about the fixed point C, is as that body and



as its distance from that point; therefore, if C be the common centre of gravity of all the bodies A, B, D, E, F, placed in the straight line AF; then is CA . A + CB . B = CD . D + CE . E + CF . F; or, the sum of the products on one side, equal to the sum of the products on the other, made by multiplying each body by its distance from that centre. And if several bodies be in equilibrium on any straight lever, then the prop is in the centre of gravity.

217. *Corol.* 5. And thought the bodies be not situated in a straight line, but scattered about in any promiscuous manner, the same property as in the last corollary still hold true, if perpendiculars to any line whatever af be drawn through



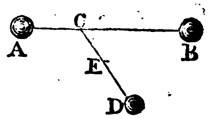
the several bodies, and their common centre of gravity, namely, that ca . A + cb = cd . D + ce . E + cf . F. For the bodies have the same effect on the line af, to turn it about the point C, whether they are placed at the points a, b, d, e, f, or in any part of the perpendiculars Aa, Bb, Dd, Ee, Ff.

PROPOSITION XL.

218. *If there be three or more Bodies, and if a Line be drawn from any one Body* D *to the Centre of Gravity of the rest* C; *then the Common Centre of Gravity* E *of all the Bodies, divides the line* CD *into two Parts in* E, *which are Reciprocally Proportional as the Body* D *to the Sum of all the other Bodies.*

That is, CE : ED : : D : A + B &c.

For, suppose the bodies A and B to be collected into the common centre of gravity C, and let their sum be called S. Then, by the last prop. CE : ED : : D : S or A + B &c.



219. *Corol.* Hence we have a method of finding the common centre of gravity of any number of bodies; namely, by first finding the centre of any two of them, then the centre of that centre and a third, and so on for a fourth, or fifth, &c.

PROPOSITION XLI.

220. *If there be taken any Point* P, *in the Line passing through the Centres of two Bodies; then the Sum of the two Products, of each Body multiplied by its Distance from that Point, is equal to the Product of the Sum of the Bodies multiplied by the Distance of their Common Centre of Gravity* C *from the same Point* P.

That is, $PA \cdot A + PB \cdot B = PC \cdot \overline{A + B}$.

For, by the 38th, $CA \cdot A = CB \cdot B$, that is, $\overline{PA - PC} \cdot A = \overline{PC - PB} \cdot B$; therefore, by adding, $PA \cdot A + PB \cdot B = PC \cdot \overline{A + B}$.

221. *Corol.* 1. Hence, the two bodies A and B have the same force to turn the lever about the point P, as if they were both placed in C their common centre of gravity.

Or, if the line, with the bodies, move about the point P; the sum of the momenta of A and B, is equal to the momentum of the sum S or A + B placed at the centre C.

222. *Corol.* 2. The same is also true of any number of bodies whatever, as will appear by cor. 4, prop. 39, namely, $PA \cdot A + PB \cdot B + PD \cdot D$ &c. $= PC \cdot \overline{A + B + D \text{ \&c}}$, where P is in any point whatever in the line AC.

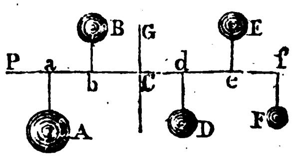
And, by cor. 5, prop. 39, the same thing is true when the bodies are not placed in that line, but any where in the perpendiculars passing through the points A, B, D, &c; namely, $Pa \cdot A + Pb \cdot B + Pd \cdot D \text{ \&c} = PC \cdot \overline{A + B + D \text{ \&c.}}$

223. *Corol.* 3. And if a plane pass through the point P perpendicular to the line CP; then the distance of the common centre of gravity from that plane, is

$$PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \text{ \&c}}{A + B + D \text{ \&c}},$$ that is, equal to the sum of all the forces divided by the sum of all the bodies. Or, if A, B, D, &c, be the several particles of one mass or compound body; then the distance of the centre of gravity of the body, below any given point P, is equal to the forces of all the particles divided by the whole mass or body, that is, equal to all the $Pa \cdot A$, $Pb \cdot B$, $Pd \cdot D$, &c, divided by the body or sum of particles A, B, D, &c.

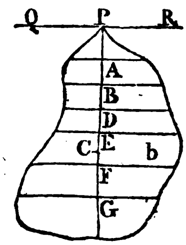
PROPOSITION XLII.

224. *To find the Centre of Gravity of any Body, or of any System of Bodies.*



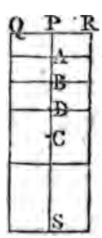
THROUGH any point P draw a plane, and let Pa, Pb, Pd, &c, be the distance of the bodies A, B, D, &c, from the plane; then, by the last cor. the distance of the common centre of gravity from the plane, will be

$$PC = \frac{Pa \cdot A + Pb \cdot B + Pd \cdot D \ \&c}{A + B + D \ \&c}.$$



225. Or, if b be any body, and QPR any plane; draw PAB &c, perpendicular to QR, and through A, B, &c, draw innumerable sections of the body b parallel to the plane QR. Let s denote any of these sections, and $d =$ PA, or PB, &c, its distance from the plane QR. Then will the distance of the centre of gravity of the body from the plane be $PC = \frac{\text{sum of all the } d\text{'s}}{b}$. And if the distance be thus found for two intersecting planes, they will give the point in which the centre is placed.

226. But the distance from one plane is sufficient for any regular body, because it is evident that, in such a figure, the centre of gravity is in the axis, or line passing through the centres of all the parallel sections.



Thus, if the figure be a parallelogram, or a cylinder, or any prism whatever; then the axis or line, or plane PS, which bisects all the sections parallel to QR, will pass through the centre of gravity of all those sections, and consequently through that of the whole figure C. Then, all the sections s being equal, and the body $b =$ PS . s, the distance of the centre will be PC =

$$\frac{PA \cdot s + PB \cdot s + \&c}{b} = \frac{PA + PB + PD \ \&c}{b} \times s = \frac{PA + PB + \&c}{PS}.$$

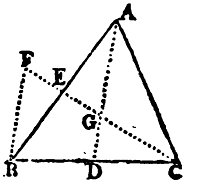
But

But PA + PB + &c, is the sum of an arithmetical progression, beginning at 0, and increasing to the greatest term PS, the number of the terms being also equal to PS; therefore the sum PA + PB + &c = $\frac{1}{2}$PS . PS; and consequently $PC = \frac{\frac{1}{2}PS \cdot PS}{PS} = \frac{1}{2}PS$; that is, the centre of gravity is in the middle of the axis of any figure whose parallel sections are equal.

227. In other figures, whose parallel sections are not equal, but varying according to some general law, it will not be easy to find the sum of all the PA . s, PB . s', PD . s'', &c, except by the general method of Fluxions; which case therefore will be best reserved, till we come to treat of that doctrine. It will be proper however to add here some examples of another method of finding the centre of gravity of a triangle, or any other right-lined plane figure.

PROPOSITION XLIII.

228. *To find the Centre of Gravity of a Triangle.*



FROM any two of the angles draw lines AD, CE, to bisect the opposite sides; so will their intersection G be the centre of gravity of the triangle.

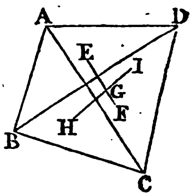
For, because AD bisects BC, it bisects also all its parallels, namely, all the parallel sections of the figure; therefore AD passes through the centres of gravity of all the parallel sections or component parts of the figure; and consequently the centre of gravity of the whole figure lies in the line AD. For the same reason, it also lies in the line CE. Consequently it is in their common point of intersection G.

229. *Corol.* The distance of the point G, is AG = $\frac{2}{3}$AD, and CG = $\frac{2}{3}$CE : or AG = 2GD, and CG = 2GE.

For, draw BF parallel to AD, and produce CE to meet it in F. Then the triangles AEG, BEF are similar, and also equal, because AE = BE; consequently AG = BF. But the triangles CDG, CBF are also equiangular, and CB being = 2CD, therefore BF = 2GD. But BF is also = AG; consequently AG = 2GD or $\frac{2}{3}$AD. In like manner, CG = 2GE or $\frac{2}{3}$CE.

PROPOSITION XLIV.

230. *To find the Centre of Gravity of a Trapezium.*



DIVIDE the trapezium ABCD into two triangles, by the diagonal BD, and find E, F, the centres of gravity of these two triangles; then shall the centre of gravity of the trapezium lie in the line EF connecting them. And therefore if EF be divided, in G, in the alternate ratio of the two triangles, namely, EG : GF :: triangle BCD : triangle ABD, then G will be the centre of gravity of the trapezium.

231. Or, having found the two points E, F, if the trapezium be divided into two other triangles BAC, DAC, by the other diagonal AC, and the centres of gravity H and I of these two triangles be also found; then the centre of gravity of the trapezium will also lie in the line HI.

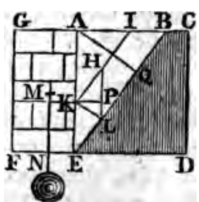
So that, lying in both the lines, EF, HI, it must necessarily lie in their intersection G.

232. And thus we are to proceed for a figure of any greater number of sides, finding the centres of their component triangles and trapeziums, and then finding the common centre of every two of these, till they be all reduced into one only.

Of the use of the place of the centre of gravity, and the nature of forces, the following practical problems are added; viz, to find the force of a bank of earth pressing against a wall, and the force of the wall to support it; also the push of an arch, with the thickness of the piers necessary to support it; also the strength and stress of beams and bars of timber and metal, &c.

PROPOSITION XLV.

233. *To determine the Force with which a Bank of Earth, or such like, presses against a Wall, and the Dimensions of the Wall necessary to Support it.*



LET ACDE be a vertical section of a bank of earth; and suppose, that if it were not supported, a triangular part of it, as ABE, would slide down, leaving it at what is called the natural slope BE; but that, by means of a wall AEFG, it is supported, and kept in its place.—It is required to find the force of ABE, ide down, and the dimensions of all AEFG, to support it.

Let

Let H be the centre of gravity of the triangle ABE, through which draw KHI parallel to the slope face of the earth BE. Now the centre of gravity H may be accounted the place of the triangle ABE, or the point into which it is all collected. Draw HL parallel, and KP perpendicular to AE, also KL perp. to IK or BE. Then if HL represent the force of the triangle ABE in its natural direction HL, HK will denote its force in its direction HK, and PK the same force in the direction PK, perpendicular to the lever EK, on which it acts. Now the three triangles EAB, HKL, HKP are all similar; therefore EB : EA :: (HL : HK ::) w the weight of the triangle EAB : $\frac{EA}{EB}w$, which will be the force of the triangle in the direction HK. Then, to find the effect of this force in the direction PK, it will be, as HK : PK :: EB : AB :: $\frac{EA}{EB}w : \frac{EA \cdot AB}{EB^2}w$, the force at K, in direction PK, perpendicularly on the lever EK, which is equal to $\frac{1}{3}$AE. But $\frac{1}{2}$AE . AB is the area of the triangle ABE; and if m be the specific gravity of the earth, then $\frac{1}{2}$AE . AB . m is as its weight. Therefore

$$\frac{EA \cdot AB}{EB^2} \cdot \tfrac{1}{2}AE \cdot AB = \frac{EA^2 \cdot AB^2}{2EB^2}m$$ is the force acting at K in direction PK. And the effect of this pressure to overturn the wall, is also as the length of the lever KE or $\frac{1}{3}$AE*: consequently

* The principle now employed in the solution of this 45th prop. is a little different from that formerly used; viz, by considering the triangle of earth ABE as acting by lines IK, &c, parallel to the face of the slope BE, instead of acting in directions parallel to the horizon AB; an alteration which gives the length of the lever EK, only the half of what it was in the former way, viz. EK $= \frac{1}{3}$AE instead of $\frac{2}{3}$AE: but every thing else remaining the same as before. Indeed this problem has formerly been treated on a variety of different hypotheses, by Mr. Muller, &c, in this country, and by many French and other authors in other countries. And this has been chiefly owing to the uncertain way in which loose earth may be supposed to act in such a case; which on account of its various circumstances of tenacity, friction, &c, will not perhaps admit of a strict mechanical certainty. On these accounts it seems probable that it is to good experiments only, made on different kinds of earth and walls, that we may probably hope for a just and satisfactory solution of the problem.

The above solution is given only in the most simple case of the problem. But the same principle may easily be extended to any other case that may be required, either in theory or practice, either with walls or banks of earth of different figures, and in different situations.

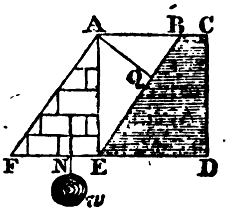
sequently its effect is $\frac{EA^3 \cdot AB^2}{6EB^2}m$, for the perpendicular force against K, to overset the wall AEFG. Which must be balanced by the counter resistance of the wall, in order that it may at least be supported.

Now, if M be the centre of gravity of the wall, into which its whole matter may be supposed to be collected, and acting in the direction MNW, its effect will be the same as if a weight W were suspended from the point N of the lever FN. Hence, if A be put for the area of the wall AEFG, and n its specific gravity; then $A \cdot n$ will be equal to the weight W, and $A \cdot n \cdot FN$ its effect on the lever to prevent it from turning about the point F. And as this effort must be equal to that of the triangle of earth, that it may just support it, which was before found equal to $\frac{EA^3 \cdot AB^2}{6EB^2}m$; therefore $A \cdot n \cdot FN = \frac{AE^3 \cdot AB^2}{6EB^2}m$, in case of an equilibrium.

234. But now, both the breadth of the wall FE, and the lever FN, or place of the centre of gravity M, will depend on the figure of the wall. If the wall be rectangular, or as broad at top as bottom; then $FN = \frac{1}{2}FE$, and the area $A = AE \cdot FE$; consequently the effort of the wall $A \cdot n \cdot FN$ is $= \frac{1}{2}FE^2 \cdot AE \cdot n$; which must be $= \frac{AE^3 \cdot AB^2}{6EB^2}m$, the effort of the earth. And the resolution of this equation gives the breadth of the wall $FE = \frac{AB \cdot AE}{EB}\sqrt{\frac{m}{3n}} = AQ\sqrt{\frac{m}{3n}}$, drawing AQ perp. to EB. So that the breadth of the wall is always proportional to the perp. depth AQ of the triangle ABE. But the breadth must be made a little more than the above value of it, that it may be more than a bare balance to the earth.—If the angle of the slope E be 45°, as it is nearly in most cases; then $FE = \frac{AE}{\sqrt{2}}\sqrt{\frac{m}{3n}} = AE\sqrt{\frac{m}{6n}} = \frac{2}{5}AE\sqrt{\frac{m}{n}}$ very nearly.

235. If the wall be of brick, its specific gravity is about 2000, and that of earth about 1984; namely, m to n as 1984 to 2000; or they may be taken as equal; then $\sqrt{\frac{m}{n}} = 1$ very nearly; and hence $FE = \frac{4}{10}AE$, or $\frac{2}{5}AE$ nearly. That is, wherever a brick rectangular wall is made to support earth, its thickness must be at least $\frac{2}{5}$ or $\frac{4}{10}$ of its height. But if the

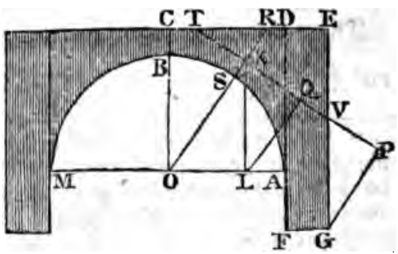
the wall be of stone, whose specific gravity is about 2520; then $\frac{m}{n} = \frac{4}{5}$, and $\sqrt{\frac{m}{n}} = \sqrt{\frac{4}{5}} = \cdot 895$; hence FE = ·358 AE $= \frac{5}{14}$AE : that is, when the rectangular wall is of stone, the breadth must be at least $\frac{5}{14}$ of its height.



236. But if the figure of the wall be a triangle, the outer side tapering to a point at top. Then the lever FN $= \frac{2}{3}$FE, and the area A $= \frac{1}{2}$FE . AE; consequently its effort A . n . FN is $= \frac{1}{3}$FE^2 . AE . n; which being put $= \frac{\text{AE}^3 \cdot \text{AB}^2}{6\text{BE}^2} m$, the equation gives FE $= \frac{\text{AB} \cdot \text{AE}}{\text{EB}} \sqrt{\frac{m}{2n}} = \text{AQ} \sqrt{\frac{m}{2n}}$ for the breadth of the wall at the bottom, for an equilibrium in this case also. —If the angle of the slope E be 45°; then will FE be $= \frac{\text{AE}}{\sqrt{2}} \sqrt{\frac{m}{2n}} = \frac{1}{2}\text{AE}\sqrt{\frac{m}{n}}$. And when this wall is of brick, then FE $= \frac{1}{2}$AE nearly. But when it is of stone; then $\frac{1}{2}\sqrt{\frac{m}{n}} = \cdot 447 = \frac{4}{9}$ nearly: that is, the triangular stone wall must have its thickness at bottom equal to $\frac{4}{9}$ of its height. And in like manner, for other figures of the wall and also for other figures of the earth.

PROPOSITION XLVI.

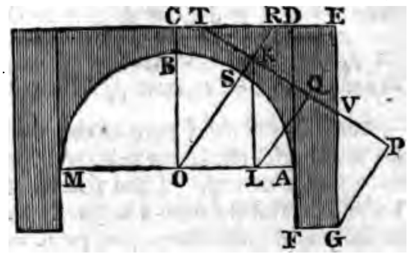
237. *To determine the Thickness of a Pier, necessary to support a Given Arch.*



LET ABCD be half the arch, and DEFG the pier. From the centre of gravity K of the half arch draw KL perp. OA; also OKR, and TKQP perp. to it; also draw LQ and GP perp. to TP, or parallel to OKR. Then if KL represent the weight of the arch BCDA, in the direction of gravity, this will resolve into KQ, the force acting against the pier perp. to the joint SR, and LQ the part of the force parallel to the same. Now KQ :

BO

notes the only force acting perp. on the arm GP, of the crooked lever FGP, to turn the pier about the point G; conseq. $KQ \times GP$ will denote the efficacious force of the arch to overturn the pier.



Again, the weight of the pier is as the area $DF \times FG$; therefore $DF . FG . \frac{1}{2}FG$, or $\frac{1}{2}DF . FG^2$, is its effect on the lever $\frac{1}{2}FG$, to prevent the pier from being overset; supposing the length of the pier, from point to point, to be no more than the thickness of the arch.

But that the pier and the arch may be in equilibrio, these two efforts must be equal. Therefore we have $\frac{1}{2}DF . FG^2 = \frac{KQ . GP . A}{KL}$, an equation, by which will be determined the thickness of the pier FG; A denoting the area of the half arch BCDA *.

Example 1. Suppose the arc ABM to be a simicircle; and that CD or OA or OB $=45$, BC $= 7$ feet, AF $=20$. Hence AD $=$ 52, DF $=$ GE $= 72$. Also by measurement are found OK $=$ 50·3, KL $=$ 40·6, LO $=$ 29·7, TD $=$ 30·87, KQ $=$ 24, the area BCDA $= 750 =$ A; and putting FG $= x$ the breadth of the pier.

Then TE $=$ TD $+$ DE $= 30·87 + x$, and KL : LO :: TE : EV $= 22·58 + 0·73x$,

then GE $-$ EV $=$ GV $= 49·42 - ·73x$,

lastly OK : KL :: GV : GP $= 39·89 - ·59x$.

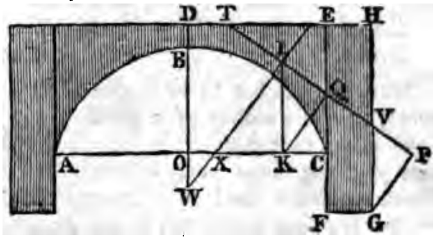
These values being now substituted in the theorem $\frac{1}{2}DF . FG^2 = \frac{KQ . GP . A}{KL}$, give $36x^2 = 17665 - 261·5x$, or $x^2 +$

* *Note*. As it is commonly a troublesome thing to calculate the place of the centre of gravity K of the half arch ADCB, it may be easily, and sufficiently near, found mechanically in the manner described in art. 211, thus: Construct that space ADCB accurately by a scale to the given dimensions, on a plate of any uniform flat substance, or even card paper; then cut it nicely out by the extreme lines, and balance it over any edge or the side of a table in two positions, and the intersection of the two places will give the situation of the point K; then the distances or lines may be measured by the scale, except those depending on the breadth of the pier FG, viz. the lines as mentioned in the examples.

7·26x

$7 \cdot 26x = 490 \cdot 7$; the root of which quadratic equation gives $x = 18 \cdot 8$ feet = DE or FG, the thickness of the pier sought.

Example 2. Suppose the span to be 100 feet, the height 40 feet, the thickness at top 6 feet, and the height of the pier to the springer 20 feet, as before.



Here the fig. may be considered as a circular segment, having the versed sine OB = 40 and the right sine OA or OC = 50; also BD = 6, CF = 20, and EF = 66. Now, by the nature of the circle, whose centre is W, the radius WB = $\frac{OB^2 + OC^2}{2OB} = \frac{40^2 + 50^2}{80} = 51\frac{1}{4}$; hence OW $= 51\frac{1}{4} - 40 = 11\frac{1}{4}$; and the area of the semi-segment OBC is found to be 1491; which taken from the rectangle ODEC = OD . OC = 46 × 50 = 2300, there remains 809 = A, the area of the space BDECB. Hence, by the method of balancing this space, and measuring the lines, there will be found, KC = 18, IK = 34·6, IX = 42, KX = 24, OX = 8, IQ = 19·4, TE = 35·6, and TH = 35·6 + x, putting x = EH, the breadth of the pier. Then IK : KX : : TH : HV = $24 \cdot 7 + 0 \cdot 7x$; hence GH − HV = $41 \cdot 3 - 0 \cdot 7x$ = GV, and IX : IK : : GV : GP = $34 \cdot 02 - 0 \cdot 58x$. These values being now substituted in the theorem $\frac{1}{2}$EF . FG^2 = $\frac{IQ . GP . A}{IK}$, gives $33x^2 = 15431 \cdot 47 - 263x$, or $x^2 + 8x = 467 \cdot 62$, the root of which quadratic equation gives $x = 18$ = EH or FG, the breadth of the pier, and which is probably very near the truth.

ON THE STRENGTH AND STRESS OF BEAMS OR BARS OF TIMBER AND METAL, &c.

238. Another use of the centre of gravity, which may be here considered, is in determining the strength and the stress of beams and bars of timber and metal, &c, in different positions; that is, the force or resistance which a beam or bar makes, to oppose any exertion or endeavour made to break it; and the force or exertion tending to break it

both of which will be different, according to the place and position of the centres of gravity.

PROPOSITION XLVII.

239. *The Absolute Strength of any Bar in the Direction of its Length, is Directly Proportional to the Area of its Transverse Section.*

Suppose the bar to be suspended by one end, and hanging freely in the manner of a pendulum; and suppose it to be strained in direction of its length, by any force, or a weight acting at the lower part, in the direction of that length, sufficient to break the bar, or to separate all its particles. Now, as the straining force acts in the direction of the length, all the particles in the transverse section of the body, where it breaks, are equally strained at the same time; and they must all separate or break together, as the bar is supposed to be of uniform texture. Thus then, the particles all adhering and resisting with equal force, the united strength of the whole, will be proportional to the number of them, or as the transverse section at the fracture.

240. *Corol.* 1. Hence the various shapes of bars make no difference in their absolute strength; this depending only on the area of the section, and must be the same in all equal areas, whether round, or square, or oblong, or solid, or hollow, &c.

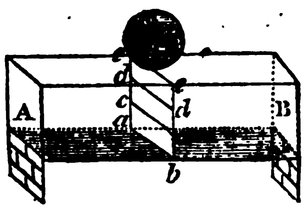
241. *Corol.* 2. Hence also, the absolute strengths of different bars, of the same materials, are to each other as their transverse sections, whatever their shape or form may be.

242. *Corol.* 3. The bar is of equal strength in every part of it, when of any uniform thickness, or prismatic shape, and is equally liable to be drawn asunder at any part of its length, whatever that length may be, by a weight acting at the bottom, independent of the weight of the bar itself; but when considered with its own weight, it is the more disposed to break, and with the less additional appended weight, the longer the bar is, on account of its own weight increasing with its length. And, for the same reason, it will be more and more liable to be broken at every point of its length, all the way in ascending or counting from the bottom to the top, where it may always be expected to part asunder. And hence we see the reason why longer bars are, in this way, more liable to break than shorter ones, or with less appended weights. Hence also we perceive that, by gradually increasing these weights, till the bar separates and breaks, then

then the last or greatest weight, is the proper measure of the absolute strength of the bar. And the same is the case with a rope, or cord, &c.—So much then for the longitudinal strength and stress of bodies. Proceed we now to consider those of their transverse actions.

PROPOSITION XLVIII.

243. *The Strength of a Beam or Bar, of Wood or Metal, &c, in a Lateral or Transverse Direction, to resist a Force acting Laterally, is Proportional to the Area or Section of the Beam in that Place, drawn into the Distance of its Centre of Gravity from the Place where the Force acts, or where the Fracture will end.*



LET AB represent the beam or bar, supported at its two ends, and on which is laid a weight W, to cause a transverse fracture *abee*. The force W acting downwards there, the fracture will commence or open across the fibres, in the opposite or lowest line *ab*; from thence, as the weight presses down the upper line *ee*, the fracture will open more and more below, and extend gradually upwards, successively to the parallel lines of fibres *cc*, *dd*, &c, till it arrive at, and finally open in the last line of fibres *ee*, where it ends; when the whole fracture is in the form of a wedge, widest at the bottom, and ending in an edge or line *ee* at top. Now the area *ae* contains and denotes the sum of all the fibres to be broken, or torn asunder; and as they are supposed to be all equal to one another, in absolute strength, that area will denote the aggregate or whole strength of all the fibres in the longitudinal direction, as in the foregoing proposition. But, with regard to lateral strength, each fibre must be considered as acting at the extremity of a lever whose centre of motion is in the line *ee*: thus, each fibre in the line *ab*, will resist the fracture, by a force proportional to the product of its individual strength into its distance *ae* from the centre of motion; consequently the resistance of all the fibres in *ab*, will be expressed by $ab \times ae$. In like manner, the aggregate resistance of another course of fibres, parallel to *ab*, as *cc*, will be denoted by $cc \times ce$; and a third, as *dd*, by $dd \times de$; and so on throughout the whole fracture. So that the sum of all these products will express the total strength or resist-

ance of all the fibres, or of the beam in that part. But, by art. 222, the sum of all these products, is equal to the product of the area *aeeb*, into the distance of its centre of gravity from *ee*. Hence the proposition is manifest.

244. *Corol.* 1. Hence it is evident that the lateral strength of a bar, must be considerably less than the absolute longitudinal strength, considered in the former proposition, and will be broken by a much less force, than was there necessary to draw the bar asunder lengthways. Because, in the one case the fibres must be all separated at once, in an instant; but in the other, they are overcome and broken successively, one after another, and in some portion of time. For instance, take a walking stick, and stretching it lengthways, it will bear a very great force before it can be drawn asunder; but again taking such a stick, apply the middle of it to the bended knee, and with the two hands drawing the ends towards you, the stick is broken across by a small force.

245. *Corol.* 2. In square beams, the lateral strengths are as the cubes of the breadths or depths.

246. *Corol.* 3. And in general, the lateral strengths of any bars, whose sections are similar figures, are as the cubes of the similar sides of the sections.

247. *Corol.* 4. In cylindrical beams, the lateral strengths are as the cubes of the diameters.

248. *Corol.* 5. In rectangular beams, the lateral strengths are to each other, as the breadths and square of the depths.

249. *Corol.* 6. Therefore a joist laid on its narrow edge, is stronger than when laid on its flat side horizontal, in proportion as the breadth exceeds the thickness. Thus, if a joist be 10 inches broad, by $2\frac{1}{2}$ thick, then it will bear 4 times more when laid on edge, than when laid flat. Which shows the propriety of the modern method of flooring, with very thin, but deep joists.

250. *Corol.* 7. If a beam be fixed firmly by one end into a wall, in a horizontal position, and the fracture be caused by a weight suspended at the other end, the process would be the same, only that the fracture would commence above, and terminate at the lower side; and the prop. and all the corollaries would still hold good.

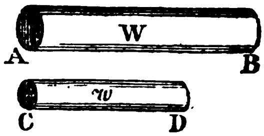
251. *Corol.* 8. When a cylinder or prism is made hollow, it is stronger than when solid, with an equal quantity of materials

rials and length, in the same proportion as its outer diameter is greater. Which shows the wisdom of Providence in making the stalks of corn, and the feathers and bones of animals, &c, to be hollow. Also, if the hollow beam have the hollow or pipe not in the middle, but nearest to that side where the fracture is to end, it will be so much the stronger.

252. *Corol.* 9. If the beam be a triangular prism, it will be strongest when laid with the edge upwards, if the fracture commence or open first on the under side; otherwise with the flat side upwards; because in either case the centre of gravity is the farther from the ending of the fracture. And the same thing is true, and for the same reason, for any other shape of the prism. On the same account also, a square beam is stronger when laid, or when acting anglewise, than when on a flat side.

PROPOSITION XLIX.

253. *The Lateral Strengths of Prismatic Beams, of the same materials, are Directly as the Areas of the Sections and the Distances of their Centres of Gravity; and Inversely as their Lengths and Weights.*



LET AB and CD represent the two beams fixed horizontally, by their ends, into an upright wall AC. Now, by the last prop. the strength of either beam, considered as without or independent of weight, is as its section drawn into the distance of its centre of gravity from the fixed point, viz. as *sc*, where *s* denotes the transverse section at A or C, and *c* the distance of its centre of gravity above the lowest point A or C. But the effort of their weight, W or *w*, tending to separate the fibres and break the beam, are, by the principle of the lever, as the weight drawn into the distance of the place where it may be supposed to be collected and applied, which is in the middle of the length of the beam; that is, the effort of the weight upon the beam is as $W \times \frac{1}{2}AB$. Hence the prop. is manifest.

254. *Corol.* 1. Any extraneous weight or force also, anywhere applied to the beam, will have a similar effect to break the beam as its own weight; that is, its effect will be as $w \times d$, as the weight drawn into the length of lever or distance from A where it is applied.

255. Cor

255. *Corol.* 2. When the beam is fixed at both ends, the same property will hold good, with this difference only, that in this case the beam is of the same strength, as another of an equal section, and only half the length, when fixed only at one end. For, if the longer beam were bisected, or cut in halves, each half would be in the same circumstances with respect to its fixed end, as the shorter beam of equal length.

256. *Corol.* 3. Square prisms and cylinders have their lateral strengths proportional to the cubes of the depths, or diameters, directly, and to their lengths and weights inversely.

Corol. 4. Similar prisms and cylinders have their strengths inversely proportional to their like linear dimensions, the smaller being comparatively larger in that proportion. For their strength increases as the cube of the diameter or of their length; but their stress, from their weight and length of lever, as the 4th power of the length.

257. *Scholium.* From the foregoing deductions it follows that, in similar bodies of the same texture, the force which tends to break them, or to make them liable to injury by accidents, in the larger bodies, increases in a higher proportion than the force which tends to preserve them entire, or to secure them against such accidents; their disadvantage, or tendency to break by their own weight, increasing in the same proportion as their length increases: so that, though a smaller beam may be firm and secure, yet a large and similar one may be so long as to break by its own weight. Hence it is justly concluded, that what may appear very firm and successful in a model or small machine, may be weak and infirm, or even fall in pieces by its own weight, when it is executed on large dimensions according to the model.

For, in similar bodies, or engines, or in animals, the greater must be always more liable to accidents than the smaller, and have a less relative strength, that is, the greater have not a strength in so great a proportion as their magnitude. A greater column, for instance, is in much more danger of breaking by a fall, than a similar smaller one. A man is in more danger from accidents of this kind than a child. An insect can bear and carry a load many times heavier than itself; whereas a large animal, as a horse, for instance, can hardly support a burden equal to his own weight.

From the same principle it is also justly inferred, that there are necessarily limits in all the works of nature and art,

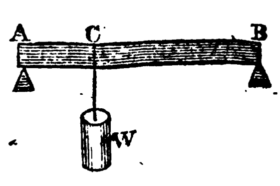
art, which they cannot surpass in magnitude. Thus, for instance, were trees to be of a very enormous size, their branches would break and fall off by their own weight. Large animals have not strength in proportion to their size: and if there were any land animals much larger than those we know, they would hardly be able to move, and would be perpetually subjected to most dangerous accidents.

As to the sea animals indeed, the case is different, as the pressure of the water in a great measure sustains them; and accordingly we find they are vastly larger than land animals.

From what has been said it clearly follows, that to make bodies, or engines, or animals, of equal relative strength, the larger ones must have grosser proportions, or a higher degree of thickness, than they have of length. And this sentiment being suggested to us by continual experience, we naturally join the idea of greater strength and force with the grosser proportions, and of agility with the more delicate ones. In architecture, where the *appearance* of solidity is no less regarded than *real* firmness and strength, in order to satisfy a judicious eye and taste, the various orders of the columns serve to suggest different ideas of strength. But, by the same principle, if we should suppose animals vastly large, from the gross proportions a heaviness and unwieldiness would arise, which would make them useless to themselves, and disagreeable to the eye. In this, as in all other cases, whatever generally pleases taste, not vitiated by prejudice of education, or by fabulous and marvellous relations, may be traced till it appears to have a just foundation in nature.

PROPOSITION L.

258. *If a Weight be placed, or a Force act, on any part of a Horizontal beam, supported at both ends, the Stress upon that part, will be as the Rectangle or Product of its two Distances from the supported ends.*



THAT is, the stress upon the beam AB, at C, by the weight W, is as AC × BC. For, by the nature of the lever, the effect of the weight W, on the lever AC, is AC . W; and the effect of this force acting at C, on the lever BC, is AC . W . BC = AC . BC . W. And, the weight W being given, the effect or stress is as AC . BC.

259. Corol.

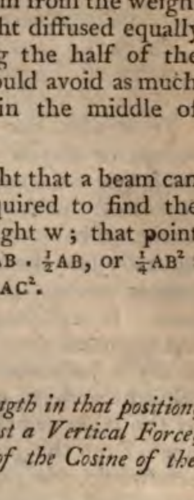
259. *Corol.* 1. The greatest stress is when the weight W is at the middle: for then the rectangle of the two halves, $AC \cdot AC = \frac{1}{2}AB \cdot \frac{1}{2}AB = \frac{1}{4}AB^2$, is the greatest. And, from the middle point, the stress is less and less all the way to the extremities A and B, where it is nothing.

260. *Corol.* 2. The same thing will obtain from the weight of the beam itself, or from any other weight diffused equally all over it; the stress in this case being the half of the former. So that, in all structures, we should avoid as much as possible, placing weights or strains in the middle of beams.

261. *Corol.* 3. If w be the greatest weight that a beam can sustain at its middle point; and it be required to find the place where it will support any greater weight W; that point will be found by making, as $W : w :: \frac{1}{2}AB \cdot \frac{1}{2}AB$, or $\frac{1}{4}AB^2 : AC \cdot BC$ or $AC \times (AB - AC) = AB \cdot AC - AC^2$.

PROPOSITION LI.

262. *When a Beam is placed aslope, its Strength in that position, is to its Strength when Horizontal, to resist a Vertical Force, as the square of Radius is to the Square of the Cosine of the Elevation.*



LET AB be the beam standing aslope, CF perp. to the horizon AFG; then CD is the vertical section of the beam, and CE, perp. to AB, is the transverse section, and is the same as when in the horizontal position. Now, the strength, in both positions, is as the section drawn into the distance of its centre of gravity from the point C. But the sections, being of the same breadth, are as their depths, CD, CE; and the distances of the centres of gravity are as the same depths; therefore the strengths are as $CD \cdot CD$ to $CE \cdot CE$, or CD^2 to CE^2. But, by the similar triangles CDE, AFD, it is $CD : CE :: AD : AF$, as radius to the cosine of the elevation. Therefore the oblique ... to the transverse strength, as AD^2 to AF^2, the ...s to the square of the cosine of elevation.

... The strength of a beam increases from the ...on, where it is least, all the way as it revolves ...position, where it is the greatest.

PROPOSITION LII.

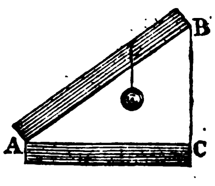
264. *When Beams stand Aslope, or Obliquely, and sustaining Weights, either at the Middle Points, or in any other Similar Situations, or Equally Diffused over their Lengths; the Strains upon them are Directly as the Weights, and the Lengths, and the Cosines of Elevation.*

For, by the inclined plane, the weight is to the pressure on the plane, as AC to AF, as radius to the cosine of elevation: therefore the pressure is as the weight drawn into the cosine of the elevation. Hence the stress will be as the length of the beam and this force; that is, as the weight × length × cosine of elevation.

265. *Corol.* 1. When the lengths and weights of beams are the same, the stress is as the cosine of elevation; and it is therefore the greatest when it lies horizontal.

266. *Corol.* 2. In all similar positions, and the weights varying as the lengths, or the beams uniform; then the stress varies as the squares of the lengths.

267. *Corol.* 3. When the weights are equal, on the oblique beam AB, and the horizontal one AC, and BC is vertical; the stress on both beams is equal. For, the length into the cosine of elevation is the same in both; or AB × cos. A = AC × radius.



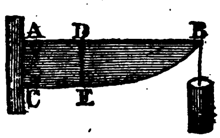
268. *Corol.* 4. But if the weights on the beams vary as their lengths; then the stress will also vary in the same ratio.

269. *Corol.* 5. And universally, the stress upon any point of an oblique beam, is as the rectangle of the segments of the beam, and the weight, and cosine of inclination, directly; and the length inversely.

PROPOSITION LIII.

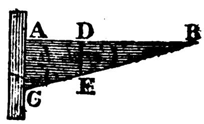
270. *When a Beam is to sustain any Weight, or Pressure, or Force, acting Laterally; then the Strength ought to be as the Stress upon it; that is, the Breadth multiplied by the Square of the Depth, or in similar sections, the Cube of the Diameter, in every place, ought to be proportional to the Length drawn into the Weight or Force acting on it. And the same is true of several Different Pieces of timber compared together.*

For every several piece of timber or metal, as well as every part of the same, ought to have its strength proportioned to the weight, force, or pressure it is to support. And therefore the strength ought to be universally, or in every part as the stress upon it. But the strength is as the breadth into the square of the depth; and the stress is as the weight or force into the distance it acts at. Therefore these must be in constant ratio. This general property will give rise to the effect of different shapes in beams, according to particular circumstances; as in the following corollaries.



271. *Corol.* 1. If ABC be a horizontal beam, fixed at the end AC, and sustaining a weight at the other end B. And if the sections at all places be similar figures; and DE be the diameter at any place D; then BD will be every where as DE^3. So that, if ADB be a right line, then BEC will be cubic parabola. In which case $\frac{2}{5}$ of such a beam may be cut away, without any diminution of the strength.—But if the beam be bounded by two parallel planes, perpendicular to the horizon; then BD will be as DE^2; and then BEC will be the common parabola. In which case a 3d part of the beam may be thus cut away.

272. *Corol.* 2. But if a weight press uniformly on every part of AB; and the sections in all points, as D, be similar; then BD^2 will be every where as DE^3: and then BEC is the semicubical parabola.



But, in this disposition of the weight, if the beam be bounded by parallel plains, perpendicular to the horizon; then BD will be always as DE; and BEC a right line, or ABC a wedge. So that then half the beam may be cut away, without diminution of strength.

273. *Corol.*

273. *Corol.* 3. If the beam AB be supported at both ends; and if it sustain a weight at any variable point D, or uniformly on all parts of its length; and if all the sections be similar figures; then will the diameter DE^3 be every where as the rectangle AD . DB.

But if it be bounded by two parallel planes, perpendicular to the horizon; then will DE^2 be every where as the rectangle AD . DB, and the curve AEB an ellipsis.

274. *Corol.* 4. But if a weight be placed at any given point F, and all the sections be similar figures; then will AD be as DE^3, and AG, BG be two cubic parabolas.

But if the beam be bounded by two parallel planes, perpendicular to the horizon; then AD is as DE^2, and AG and BG are two common parabolas.

275. *Scholium.* The relative strengths of several sorts of wood, and of other bodies, as determined by Mr. Emerson, are as follow:

Iron	107
Brass	50
Bone	22
Box, Yew, Plumtree, Oak	11
Elm, Ash	$8\frac{1}{2}$
Walnut, Thorn	$7\frac{1}{2}$
Red fir, Holly, Elder, Plane, Crabtree, Appletree	7
Beech, Cherrytree, Hazle	$6\frac{2}{3}$
Lead	$6\frac{1}{2}$
Alder, Asp, Birch, White fir, Willow	6
Fine freestone	1

A cylindric rod of good clean fir, of 1 inch circumference, drawn lengthways, will bear at extremity 400 lbs; and a spear of fir, 2 inches diameter, will bear about 7 tons in that direction.

A rod of good iron, of an inch circumference, will bear a stretch of near 3 tons weight.

A good hempen rope, of an inch circumference, will bear 1000 lbs at the most.

Hence Mr. Emerson concludes, that if a rod of fir, or of iron,

iron, or a rope of d inches diameter, were to lift $\frac{1}{4}$ of the extreme weight; then

The fir would bear $8\frac{4}{5}\ d^2$ hundred weights.
The rope - - $22\ d^2$ ditto.
The iron - - $6\frac{3}{4}\ d^2$ tons.

Mr. Banks, an ingenious lecturer on mechanics, made many experiments on the strength of wood and metal; whence he concludes, that cast iron is from $3\frac{1}{2}$ to $4\frac{1}{2}$ times stronger than oak of equal dimensions; and from 5 to $6\frac{1}{2}$ times stronger than deal. And that bars of cast iron, an inch square, weighing 9 lbs. to the yard in length, supported at the extremities, bear on an average, a load of 970 lbs. laterally. And they bend about an inch before they break.

Many other experiments on the strength of different materials, and curious results deduced from them, may be seen in Dr. Gregory's and Mr. Emerson's Treatises on Mechanics, as well as some more propositions on the strength and stress of different bars.

On the CENTRES of PERCUSSION, OSCILLATION, and GYRATION.

276. THE Centre of Percussion of a body, or a system of bodies, revolving about a point, or axis, is that point, which striking an immoveable object, the whole mass shall not incline to either side, but rest as it were in equilibrio, without acting on the centre of suspension.

277. The Centre of Oscillation is that point, in a body vibrating by its gravity, in which if any body be placed, or if the whole mass be collected, it will perform its vibrations in the same time, and with the same angular velocity, as the whole body, about the same point or axis of suspension.

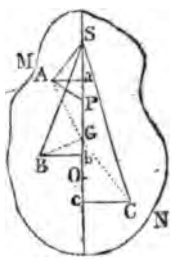
278. The Centre of Gyration, is that point, in which if the whole mass be collected, the same angular velocity will be generated in the same time, by a given force acting at any place, as in the body or system itself.

279. The angular motion of a body, or system of bodies, is the motion of a line connecting any point and the centre or axis of motion; and is the same in all parts of the same revolving body. And in different unconnected bodies, each revolving about a centre, the angular velocity is as the absolute velocity directly, and as the distance from the centre inversely; so that, if their absolute velocities be as their radii or distances, the angular velocities will be equal.

PRO-

PROPOSITION LIV.

280. *To find the Centre of Percussion of a Body, or System of Bodies.*



LET the body revolve about an axis passing through any point S in the line SGO, passing through the centres of gravity and percussion, G and O. Let MN be the section of the body, or the plane in which the axis SGO moves. And conceive all the particles of the body to be reduced to this plane, by perpendiculars let fall from them to the plane: a supposition which will not affect the centres G, O, nor the angular motion of the body.

Let A be the place of one of the particles, so reduced; join SA, and draw AP perpendicular to AS, and Aa perpendicular to SGO: then AP will be the direction of A's motion as it revolves about S; and the whole mass being stopped at O, the body A will urge the point P, forward, with a force proportional to its quantity of matter and velocity, or to its matter and distance from the point of suspension S; that is, as A . SA; and the efficacy of this force in a direction perpendicular to SO, at the point P, is as A . Sa, by similar triangles; also, the effect of this force on the lever, to turn it about O, being as the length of the lever, is as A . Sa . PO = A . Sa . (SO − SP) = A . Sa . SO − A . Sa . SP = A . Sa . SO − A . SA². In like manner, the forces of B and C, to turn the system about O, are as

$$B . Sb . SO - B . SB^2, \text{ and}$$
$$C . Sc . SO - C . SC^2, \&c.$$

But, since the forces on the contrary sides of O destroy one another, by the definition of this force, the sum of the positive parts of these quantities must be equal to the sum of the negative parts,

that is, $A . Sa . SO + B . Sb . SO + C . Sc . SO \ \&c = A . SA^2 + B . SB^2 + C . SC^2 \ \&c$; and

$$\text{hence } SO = \frac{A . SA^2 + B . SB^2 + C . SC^2 \ \&c}{A . Sa + B . Sb + C . Sc \ \&c},$$

which is the distance of the centre of percussion below the axis of motion.

And here it must be observed that, if any of the points a, b, &c, fall on the contrary side of s, the corresponding product A . sa, or B . sb, &c, must be made negative.

281. *Corol.* 1. Since, by cor. 3, pr. 40, A + B + C &c, or the body b × the distance of the centre of gravity, SG, is = A . sa + B . sb + C . sc &c, which is the denominator of the value of SO; therefore the distance of the centre of percussion, is $\text{SO} = \dfrac{\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2 + \text{C} \cdot \text{SC}^2 \text{ \&c}}{\text{SG} \times \text{body } b}$.

282. *Corol.* 2. Since, by Geometry, theor. 36, 37,

it is $\text{SA}^2 = \text{SG}^2 + \text{GA}^2 - 2\text{SG} \cdot \text{Ga}$,

and $\text{SB}^2 = \text{SG}^2 + \text{GB}^2 + 2\text{SG} \cdot \text{Gb}$,

and $\text{SC}^2 = \text{SG}^2 + \text{GC}^2 + 2\text{SG} \cdot \text{GC}$, &c;

and, by cor. 5, pr. 40, the sum of the last terms is nothing, namely, $-\ 2\text{SG} \cdot \text{Ga} + 2\text{SG} \cdot \text{Gb} + 2\text{SG} \cdot \text{GC} \text{ \&c} = 0$; therefore the sum of the others, or $\text{A} \cdot \text{SA}^2 + \text{B} \cdot \text{SB}^2$ &c is $= (\text{A} + \text{B} \text{ \&c}) \cdot \text{SG}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2$ &c, or $= b \cdot \text{SG}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2$ &c; which being substituted in the numerator of the foregoing value of SO, gives

$$\text{SO} = \frac{b \cdot \text{SG}^2 + \text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{\&c}}{b \cdot \text{SG}},$$

$$\text{or SO} = \text{SG} + \frac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 + \text{C} \cdot \text{GC}^2 \text{ \&c}}{b \cdot \text{SG}}.$$

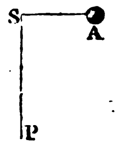
283. *Corol.* 3. Hence the distance of the centre of percussion always exceeds the distance of the centre of gravity, and the excess is always $\text{GO} = \dfrac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 \text{ \&c}}{b \cdot \text{SG}}$.

284. And hence also, $\text{SG} \cdot \text{GO} = \dfrac{\text{A} \cdot \text{GA}^2 + \text{B} \cdot \text{GB}^2 \text{ \&c}}{\text{the body } b}$; that is SG . GO is always the same constant quantity, wherever the point of suspension S is placed; since the point G and the bodies A, B, &c, are constant. Or GO is always reciprocally as SG, that is GO is less, as SG is greater; and consequently the point O rises upwards and approaches towards the point G, as the point S is removed to the greater distance; and they coincide when SG is infinite. But when with G, then GO is infinite, or O is at an infinite

PROPO-

PROPOSITION LV.

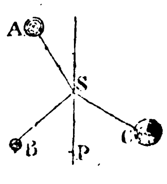
285. *If a Body* A, *at the Distance* SA *from an axis passing through* S, *be made to revolve about that axis by any Force acting at* P *in the Line* SP, *Perpendicular to the Axis of Motion: It is required to determine the Quantity or Matter of another Body* Q, *which being placed at* P, *the Point where the Force acts, it shall be accelerated in the Same Manner, as when* A *revolved at the Distance* SA; *and consequently, that the Angular Velocity of* A *and* Q *about* S, *may be the Same in Both Cases.*



By the nature of the lever, SA : SP : : f : $\frac{SP}{SA} \cdot f$, the effect of the force f, acting at P, on the body at A; that is, the force f acting at P, will have the same effect on the body A, as the force $\frac{SP}{SA} f$, acting directly at the point A. But as ASP revolves altogether about the axis at S, the absolute velocities of the points A and S, or of the bodies A and Q, will be as the radii SA, SP, of the circle described by them. Here then we have two bodies A and Q, which being urged directly by the forces f and $\frac{SP}{SA} f$, acquire velocities which are as SP and SA. And since the motive forces of bodies are as their mass and velocity: therefore - - - -

$\frac{SP}{SA} f : f : : A \cdot SA : Q \cdot SP$, and $SP^2 : SA^2 : : A : Q = \frac{SA^2}{SP^2} A$, which therefore expresses the mass of matter which, being placed at P, would receive the same angular motion from the action of any force at P, as the body A receives. So that the resistance of any body A, to a force acting at any point P, is directly as the square of its distance SA from the axis of motion, and reciprocally as the square of the distance SP of the point where the force acts.

286. *Corol.* 1. Hence the force which accelerates the point P, is to the force of gravity, as $\frac{f \cdot SP^2}{A \cdot SA^2}$ to 1, or as $f \cdot SP^2$ to $A \cdot SA^2$.



287. *Corol.* 2. If any number of bodies A, B, C, be put in motion, about a fixed axis passing through S, by a force acting at P; the point P will be accelerated in the same manner, and consequently the whole system will have the same angular velocity, if instead of the

bodies

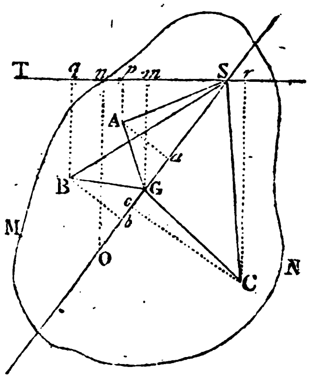
bodies A, B, C, placed at the distances SA, SB, SC, there be substituted the bodies $\frac{SA^2}{SP^2}A$, $\frac{SB^2}{SP^2}B$, $\frac{SC^2}{SP^2}C$; these being collected into the point P. And hence, the moving force being f, and the matter moved being $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$; theref. $\frac{f \cdot SP^2}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$ is the accelerating force; which therefore is to the accelerating force of gravity, as $f \cdot SP^2$ to $A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$.

288. *Corol.* 3. The angular velocity of the whole system of bodies, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$. For the absolute velocity of the point P, is as the accelerating force, or directly as the motive force f, and inversely as the mass $\frac{A \cdot SA^2 \ \&c}{SP^2}$: but the angular velocity is as the absolute velocity directly, and the radius SP inversely; therefore the angular velocity of P, or of the whole system, which is the same thing, is as $\frac{f \cdot SB}{A \cdot SA^2 + B \cdot SB^2 + C' \cdot SC^2}$.

PROPOSITION LVI.

289. *To determine the Centre of Oscillation of any Compound Mass, or Body* MN, *or of any System of Bodies* A, B, C, *&c.*

LET MN be the plane of vibration, to which let all the matter be reduced, by letting fall perpendiculars from every particle, to this plane. Let G be the centre of gravity, and O the centre of oscillation; through the axis S draw SGO, and the horizontal line sq; then from every particle A, B, C, &c, let fall perpendiculars Aa, Ap, Bb, Bq, Cc, Cr, to these two lines; and join SA, SB, SC; also, draw Gm, On, perpendicular to sq. Now the forces of the weights A, B, C, to turn the body about the axis, are A . sp, B . sq, + C . sr; therefore, by cor. 3, prop. 55, the angular



motion

motion generated by all these forces is $\frac{A \cdot sp + B \cdot sq - C \cdot sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$. Also, the angular veloc. any particle p, placed in O, generates in the system, by its weight, is $\frac{p \cdot sn}{p \cdot SO^2}$ or $\frac{sn}{SO^2}$, or $\frac{sm}{SG \cdot SO}$, because of the similar triangles SGm, SOn. But, by the problem, the vibrations are performed alike in both cases, and therefore these two expressions must be equal to each other, that is, $\frac{sm}{SG \cdot SO} = \frac{A \cdot sp + B \cdot sq - C \cdot sr}{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}$; and hence $SO = \frac{sm}{SG} \times \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A \cdot sp + B \cdot sq - C \cdot sr}$.

But, by cor. 2, pr. 41, the sum $A \cdot sp + B \cdot sq - C \cdot sr = (A + B + C) \cdot sm$; therefore the distance $SO =$ - - - $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SG \cdot (A + B + C)} = \frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A \cdot Sa + B \cdot Sb + C \cdot Sc}$ by prop. 42, which is the distance of the centre of oscillation O, below the axis of suspension; where any of the products $A \cdot Sa$, $B \cdot Sb$, must be negative, when a, b, &c, lie on the other side of S. So that this is the same expression as that for the distance of the centre of percussion, found in prop. 54.

Hence it appears, that the centres of percussion and of oscillation, are in the very same point. And therefore the properties in all the corollaries there found for the former, are to be here understood of the latter.

290. *Corol.* 1. If p be any particle of a body b, and d its distance from the axis of motion S; also G, O the centres of gravity and oscillation. Then the distance of the centre of oscillation of the body, from the axis of motion, is - - - $SO = \frac{\text{sum of all the } pd^2}{SG \times \text{the body } b}$.

291. *Corol.* 2. If b denote the matter in any compound body, whose centres of gravity and oscillation are G and O; the body P, which being placed at P, where the force acts as in the last proposition, and which receives the same motion from that force as the compound body b, is $P = \frac{SG \cdot SO}{SP^2} \cdot b$.

For, by corol. 2, prop. 54, this body P is = - - - - $\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$. But, by corol. 1, prop. 53,

SG

$SG \cdot SO \cdot b = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$; therefore $P = \frac{SG \cdot SO}{SP} \cdot b$.

SCHOLIUM.

292. By the method of Fluxions, the centre of oscillation, for a regular body, will be found from cor. 1. But for an irregular one; suspend it at the given point; and hang up also a simple pendulum of such a length, that making them both vibrate, they may keep time together. Then the length of the simple pendulum, is equal to the distance of the centre of oscillation of the body, below the point of suspension.

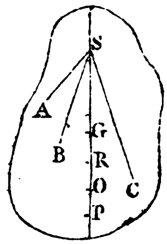
293. Or it will be still better found thus: Suspend the body very freely by the given point, and make it vibrate in small arcs, counting the number of vibrations it makes in any time, as a minute, by a good stop watch; and let that number of vibrations made in a minute be called n: Then shall the distance of the centre of oscillation, be $SO = \frac{140850}{nn}$ inches. For, the length of the pendulum vibrating seconds, or 60 times in a minute, being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the square of the number of vibrations made in the same time; therefore - - -
$n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{nn} = \frac{140850}{nn}$: the length of the pendulum which vibrates n times in a minute, or the distance of the centre of oscillation below the axis of motion.

294. The foregoing determination of the point, into which all the matter of a body being collected, it shall oscillate in the same manner as before, only respects the case in which the body is put in motion by the gravity of its own particles, and the point is the centre of oscillation: but when the body is put in motion by some other extraneous force, instead of its gravity, then the point is different from the former, and is called the Centre of Gyration; which is determined in the following manner:

PRO-

PROPOSITION LVII.

295. *To determine the Centre of Gyration of a Compound Body or of a System of Bodies.*



LET R be the centre of gyration, or the point into which all the particles A, B, C, &c, being collected, it shall receive the same angular motion from a force f acting at P, as the whole system receives.

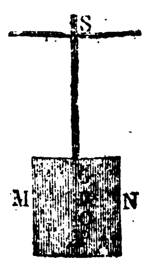
Now, by cor. 3, pr. 54, the angular velocity generated in the system by the force f, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 \&c}$, and by the same, the angular velocity of the system placed in R, is $\frac{f \cdot SP}{(A + B + C \&c) \cdot SR^2}$: then, by making these two expressions equal to each other, the equation gives - - - $SR = \sqrt{\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A + B + C}}$, for the distance of the centre of gyration below the axis of motion.

296. *Corol.* 1. Because $A \cdot SA^2 + B \cdot SB^2$ &c $= SG \cdot SO \cdot b$, where G is the centre of gravity, O the centre of oscillation, and b the body $A + B + C$ &c; therefore $SR^2 = SG \cdot SO$; that is, the distance of the centre of gyration, is a mean proportional between those of gravity and oscillation.

297. *Corol.* 2. If p denote any particle of a body b, at d distance from the axis of motion; then $SR^2 = \frac{\text{sum of all the } pd^2}{\text{body } b}$.

PROPOSITION LVIII.

298. *To determine the Velocity with which a Ball moves, which being shot against a Ballistic Pendulum, causes it to vibrate through a given Angle.*



THE Ballistic Pendulum is a heavy block of wood MN, suspended vertically by a strong horizontal iron axis at S, to which it is connected by a firm iron stem. This problem is the application of the last proposition, or of prop. 54, and was invented by the very ingenious Mr. Robins, to determine the initial velocities of military projectiles; a circumstance very useful in that science; and it is the best method yet known for determining them with any degree of accuracy.

Let

Let G, R, O be the centres of gravity, gyration, and oscillation, as determined by the foregoing propositions; and let P be the point where the ball strikes the face of the pendulum; the momentum of which, or the product of its weight and velocity, is expressed by the force f, acting at P, in the foregoing propositions. Now,

Put p = the whole weight of the pendul.
b = the weight of the ball,
g = SG the dist. of the cen. of grav.
o = SO the dist. of the cen. of oscilla.
r = SR = $\sqrt{go}$ the dist. of cen. of gyr.
i = SP the dist. of the point of impact,
v = the velocity of the ball,
u = that of the point of impact P,
c = chord of the arc described by O.

By prop. 56, if the mass p be placed all at R, the pendulum will receive the same motion from the blow in the point P: and as $SP^2 : SR^2 :: p : \frac{SR^2}{SP^2} . p$ or $\frac{r^2}{i^2} p$ or $\frac{go}{ii} p$, (prop. 54), the mass which being placed at P, the pendulum will still receive the same motion as before. Here then are two quantities of matter, namely, b and $\frac{go}{ii} p$, the former moving with the velocity v, and striking the latter at rest; to determine their common velocity u, with which they will jointly proceed forward together after the stroke. In which case, by the law of the impact of non-elastic bodies, we have $\frac{go}{ii} p + b : b :: v : u$, and therefore $v = \frac{bii + gop}{bii} u$ the velocity of the ball in terms of u, the velocity of the point P, and the known dimensions and weights of the bodies.

But now to determine the value of u, we must have recourse to the angle through which the pendulum vibrates; for when the pendulum descends down again to the vertical position, it will have acquired the same velocity with which it began to ascend, and, by the laws of falling bodies, the velocity of the centre of oscillation is such, as a heavy body would acquire by freely falling through the versed sine of the arc described by the same centre O. But the chord of that arc is c, and its radius is o; and, by the nature of the circle, the chord is a mean proportional between the versed sine and diameter, therefore $2o : c :: c : \frac{cc}{2o}$, the versed sine of the arc described by O. Then, by the laws of falling bodies

$\sqrt{16}$

$\sqrt{16\tfrac{1}{12}} : \sqrt{\dfrac{cc}{2o}} :: 32\tfrac{1}{6} : c\sqrt{\dfrac{2a}{o}}$, the velocity acquired by the point o in descending through the arc whose chord is c, where $a = 16\tfrac{1}{12}$ feet: and therefore $o : i :: c\sqrt{\dfrac{2a}{o}} : \dfrac{ci}{o}\sqrt{\dfrac{2a}{o}}$, which is the velocity u, of the point P.

Then, by substituting this value for u, the velocity of the ball, before found, becomes $v = \dfrac{bii + gop}{bio} \times c\sqrt{\dfrac{2a}{o}}$. So that the velocity of the ball is directly as the chord of the arc described by the pendulum in its vibration.

SCHOLIUM.

299. In the foregoing solution, the change in the centre of oscillation is omitted, which is caused by the ball lodging in the point P. But the allowance for that small change, and that of some other small quantities, may be seen in my Tracts, where all the circumstances of this method are treated at full length.

300. For an example in numbers of this method, suppose the weights and dimensions to be as follow: namely,

$p = 570$lb,
$b = 18$oz $1\tfrac{1}{2}$dr $= 1{\cdot}131$lb,
$g = 78\tfrac{1}{2}$ inc.
$o = 84\tfrac{7}{9}$ inc. $= 7{\cdot}065$ feet
$i = 94\tfrac{3}{10}$ inc.
$c = 18{\cdot}73$ inc.

Then

$$\frac{bii+gop}{bio} \times c = \frac{1{\cdot}131 \times 94{\cdot}3^2 + 78\tfrac{1}{2} \times 84\tfrac{7}{9} \times 570}{1{\cdot}131 \times 94\tfrac{3}{10} \times 84\tfrac{7}{9}} \times \frac{18{\cdot}73}{12} = 656{\cdot}56,$$

And $\sqrt{\dfrac{2a}{o}} = \sqrt{\dfrac{32\tfrac{1}{6}}{7{\cdot}065}} = \sqrt{\dfrac{193}{42{\cdot}39}} = 2{\cdot}1337$.

Therefore $656{\cdot}56 \times 2{\cdot}1337$, or 1401 feet, is the velocity, per second, with which the ball moved when it struck the pendulum.

OF HYDROSTATICS.

301. HYDROSTATICS is the science which treats of the pressure, or weight, and equilibrium of water and other fluids, especially those that are non-elastic.

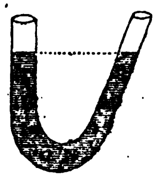
302. A fluid is elastic, when it can be reduced into a less volume by compression, and which restores itself to its former bulk again when the pressure is removed; as air. And it is non-elastic, when it is not compressible by such force; as water, &c.

PROPOSITION LIX.

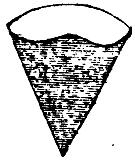
303. *If any Part of a Fluid be raised higher than the rest, by any Force, and then left to itself; the higher Parts will descend to the lower Places, and the Fluid will not rest, till its Surface be quite even and level.*

For, the parts of a fluid being easily moveable every way, the higher parts will descend by their superior gravity, and raise the lower parts, till the whole come to rest in a level or horizontal plane.

304. *Corol.* 1. Hence, water that communicates with other water, by means of a close canal or pipe, will stand at the same height in both places. Like as water in the two legs of a syphon.



305. *Corol.* 2. For the same reason, if a fluid gravitate towards a centre; it will dispose itself into a spherical figure, the centre of which is the centre of force. Like the sea in respect of the earth.



PROPOSITION LX.

306. *When a Fluid is at Rest in a Vessel, the Base of which is Parallel to the Horizon; Equal Parts of the Base are Equally Pressed by the Fluid.*

For, on every equal part of this base there is an equal column of the fluid supported by it. And as all the columns are of equal height, by the last proposition they are of equal weight, and therefore they press the base equally; that is, equal parts of the base sustain an equal pressure.

307. *Corol.* 1. All parts of the fluid press equally at the same depth. For, if a plane parallel to the horizon be conceived to be drawn at that depth; then the pressure being the same in any part of that plane, by the proposition, therefore the parts of the fluid, instead of the plane, sustain the same pressure at the same depth.

308. *Corol.* 2. The pressure of the fluid at any depth, is as the depth of the fluid. For the pressure is as the weight, and the weight is as the height of the fluid.

309. *Corol.*

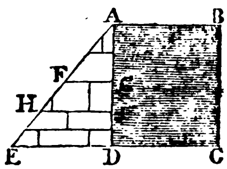
309. *Corol.* 3. The pressure of the fluid on any horizontal surface or plane, is equal to the weight of a column of the fluid, whose base is equal to that plane, and altitude is its depth below the upper surface of the fluid.

PROPOSITION LXI.

310. *When a Fluid is Pressed by its own Weight, or by any other Force; at any Point it Presses Equally, in all Directions whatever.*

THIS arises from the nature of fluidity, by which it yields to any force in any direction. If it cannot recede from any force applied, it will press against other parts of the fluid in the direction of that force. And the pressure in all directions will be the same: for if it were less in any part, the fluid would move that way, till the pressure be equal every way.

311. *Corol.* 1. In a vessel containing a fluid; the pressure is the same against the bottom, as against the sides, or even upwards at the same depth.



312. *Corol.* 2. Hence, and from the last proposition, if ABCD be a vessel of water, and there be taken, in the base produced, DE, to represent the pressure at the bottom; joining AE, and drawing any parallels to the base, as FG, HI; then shall FG represent the pressure at the depth AG, and HI the pressure at the depth AI, and so on; because the parallels - FG, HI, ED, by sim. triangles, are as the depths AG, AI, AD: which are as the pressures, by the proposition.

And hence the sum of all the FG, HI, &c, or area of the triangle ADE, is as the pressure against all the points G, I, &c, that is, against the line AD. But as every point in the line CD is pressed with a force as DE, and that thence the pressure on the whole line CD is as the rectangle ED . DC, while that against the side is as the triangle ADE or $\frac{1}{2}$AD . DE; therefore the pressure on the horizontal line DC, is to the pressure against the vertical line DA, as DC to $\frac{1}{2}$DA. And hence, if the vessel be an upright rectangular one, the pressure on the bottom, or whole weight of the fluid, is to the pressure against one side, as the base is to half that side. Therefore the weight of the fluid is to the pressure against

all

all the four upright sides, as the base is to half the upright surface. And the same holds true also in any upright vessel, whatever the sides be, or in a cylindrical vessel. Or, in the cylinder, the weight of the fluid, is to the pressure against the upright surface, as the radius of the base is to double the altitude.

Also, when the rectangular prism becomes a cube, it appears that the weight of the fluid on the base, is double the pressure against one of the upright sides, or half the pressure against the whole upright surface.

313. *Corol.* 3. The pressure of a fluid against any upright surface, as the gate of a sluice or canal, is equal to half the weight of a column of the fluid whose base is equal to the surface pressed, and its altitude the same as the altitude of that surface. For the pressure on a horizontal base equal to the upright surface, is equal to that column; and the pressure on the upright surface, is but half that on the base, of the same area.

So that, if b denote the breadth, and d the depth of such a gate or upright surface; then the pressure against it, is equal to the weight of the fluid whose magnitude is $\frac{1}{2}bd^2 = \frac{1}{2}\text{AB} \cdot \text{AD}^2$. Hence, if the fluid be water, a cubic foot of which weighs 1000 ounces, or $62\frac{1}{2}$ pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or $\frac{1}{2}\text{AB} \cdot \text{AD}^2$, is 1440 feet; and the pressure is 1440000 ounces, or 90000 pounds, or $40\frac{1}{5}$ tons weight nearly.

PROPOSITION LXII.

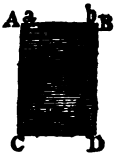
314. *The pressure of a Fluid on a Surface any how immersed in it, either Perpendicular, or Horizontal, or Oblique; is Equal to the Weight of a Column of the Fluid, whose Base is equal to the Surface pressed, and its Altitude equal to the Depth of the Centre of Gravity of the Surface pressed below the Top or Surface of the Fluid.*

For, conceive the surface pressed to be divided into innumerable sections parallel to the horizon; and let s denote any one of those horizontal sections, also d its distance or depth below the top surface of the fluid. Then, by art. 309, the pressure of the fluid on the section is equal to the weight of ds; consequently the total pressure on the whole surface is equal to all the weights ds. But, if b denote the whole surface pressed, and g the depth of its centre of gravity below the top of the fluid; then, by art. 256 or 259, bg is equal to

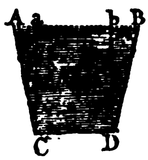
to the sum of all the *ds*. Consequently the whole pressure of the fluid on the body or surface *b*, is equal to the weight of the bulk *bg* of the fluid, that is, of the column whose base is the given surface *b*, and its height is *g* the depth of the centre of gravity in the fluid.

PROPOSITION LXIII.

315. *The Pressure of a Fluid, on the Base of the Vessel in which it is contained, is as the Base and Perpendicular Altitude; whatever be the Figure of the Vessel that contains it.*



If the sides of the base be upright, so that it be a prism of a uniform width throughout; then the case is evident; for then the base supports the whole fluid, and the pressure is just equal to the weight of the fluid.



But if the vessel be wider at top than bottom; then the bottom sustains, or is pressed by, only the part contained within the upright lines ac, bD; because the parts ACa, BDb are supported by the sides AC, BD; and those parts have no other effect on the part abDC than keeping it in its position, by the lateral pressure against ac and bD, which does not alter its perpendicular pressure downwards. And thus the pressure on the bottom is less than the weight of the contained fluid.

And if the vessel be widest at bottom; then the bottom is still pressed with a weight which is equal to that of the whole upright column abDC. For, as the parts of the fluid are in equilibrio, all the parts have an equal pressure at the same depth; so that the parts within Cc and dD press equally as those in cd, and therefore equally the same as if the sides of the vessel had gone upright to a and b, the defect of fluid in the parts ACa and BD*b* being exactly compensated by the downward pressure or resistance of the sides AC and BD against the contiguous fluid. And thus the pressure on the base may be made to exceed the weight of the contained fluid, in any proportion whatever.

So that, in general, be the vessels of any figure whatever, regular or irregular, upright or sloping, or variously wide and narrow in different parts, if the bases and perpendicular altitudes be but equal, the bases always sustain the same pressure. And as that pressure, in the regular upright

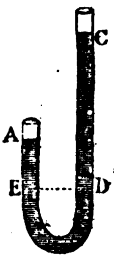
 vessel,

vessel, is the whole column of the fluid, which is as the base and altitude; therefore the pressure in all figures is in that same ratio.

316. *Corol.* 1. Hence, when the heights are equal, the pressures are as the bases. And when the bases are equal, the pressure is as the height. But when both the heights and bases are equal, the pressures are equal in all, though their contents be ever so different.

317. *Corol.* 2. The pressure on the base of any vessel, is the same as on that of a cylinder, of an equal base and height.

318. *Corol.* 3. If there be an inverted syphon, or bent tube, ABC, containing two different fluids CD, ABD, that balance each other, or rest in equilibrio; then their heights in the two legs, AE, CD, above the point of meeting, will be reciprocally as their densities.

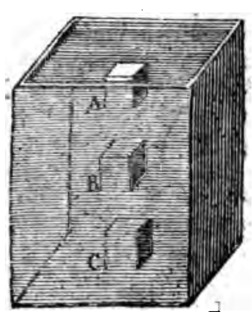


For if they do not meet at the bottom, the part BD balances the part BE, and therefore the part CD balances the part AE; that is, the weight of CD is equal to the weight of AE. And as the surface at D is the same, where they act against each other, therefore AE : CD : : density of CD : density of AE.

So, if CD be water, and AE quicksilver, which is near 14 times heavier; then CD will be = 14AE; that is, if AE be 1 inch, CD will be 14 inches; if AE be 2 inches, CD will be 28 inches; and so on.

PROPOSITION LXIV.

319. *If a Body be Immersed in a Fluid of the Same Density or Specific Gravity; it will Rest in any Place where it is put. But a Body of Greater Density will Sink; and one of a Less Density will Rise to the Top, and Float.*



THE body, being of the same density, or of the same weight with the like bulk of the fluid, will press the fluid under it, just as much as if its space was filled with the fluid itself. The pressure then all around it will be the same as if the fluid were in its place; consequently there is no force, neither upward nor downward, to put the body out of its place. And therefore it will remain wherever it is put.

But

But if the body be lighter; its pressure downward will be less than before, and less than the water upward at the same depth; therefore the great force will overcome the less, and push the body upward to A.

And if the body be heavier than the fluid, the pressure downward will be greater than the fluid at the same depth; therefore the greater force will prevail, and carry the body down to the bottom at C.

320. *Corol.* 1. A body immersed in a fluid, loses as much weight, as an equal bulk of the fluid weighs. And the fluid gains the same weight. Thus, if the body be of equal density with the fluid, it loses all its weight, and so requires no force but the fluid to sustain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the same bulk of water; and it requires a force to sustain it just equal to that difference. But if it be lighter, it requires a force equal to the same difference of weights to keep it from rising up in the fluid.

321. *Corol.* 2. The weights lost, by immerging the same body in different fluids, are as the specific gravities of the fluids. And bodies of equal weight, but different bulk, lose, in the same fluid, weights which are reciprocally as the specific gravities of the bodies, or directly as their bulks.

322. *Corol.* 3. The whole weight of a body which will float in a fluid, is equal to as much of the fluid, as the immersed part of the body takes up, when it floats. For the pressure under the floating body, is just the same as so much of the fluid as is equal to the immersed part; and therefore the weights are the same.

323. *Corol.* 4. Hence the magnitude of the whole body, is to the magnitude of the part immersed, as the specific gravity of the fluid, is to that of the body. For, in bodies of equal weight, the densities, or specific gravities, are reciprocally as their magnitudes.

324. *Corol.* 5. And because, when the weight of a body taken in a fluid, is subtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the specific gravity of the fluid, is to that of body.

Therefore, if W be the weight of a body in air,
w its weight in water, or any fluid,
S the specific gravity of the body, and
s the specific gravity of the fluid;

 then

then $W - w : W :: s : S$, which proportion will give either of those specific gravities, the one from the other.

Thus $S = \frac{W}{W - w} s$, the specific gravity of the body;

and $s = \frac{W - w}{W} S$, the specific gravity of the fluid.

So that the specific gravities of bodies, are as their weights in the air directly, and their loss in the same fluid inversely.

325. *Corol.* 6. And hence, for two bodies connected together, or mixed together into one compound, of different specific gravities, we have the following equations, denoting their weights and specific gravities, as below, viz.

$\left.\begin{array}{l} \text{H} = \text{weight of the heavier body in air,} \\ h = \text{weight of the same in water,} \end{array}\right\}$ S its spec. gravity;
$\left.\begin{array}{l} \text{L} = \text{weight of the lighter body in air,} \\ l = \text{weight of the same in water,} \end{array}\right\}$ s its spec. gravity;
$\left.\begin{array}{l} \text{C} = \text{weight of the compound in air,} \\ c = \text{weight of the same in water,} \end{array}\right\}$ f its spec. gravity;
w = the specific gravity of water. Then,

1st, $(\text{H} - h)\,\text{S} = \text{H}w$,
2d, $(\text{L} - l)\,s = \text{L}w$,
3d, $(\text{C} - c)\,f = \text{C}w$,
4th, $\text{H} + \text{L} = \text{C}$,
5th, $h + l = c$,
6th, $\frac{\text{H}}{\text{S}} + \frac{\text{L}}{s} = \frac{\text{C}}{f}$

From which equations may be found any of the above quantities, in terms of the rest.

Thus, from one of the first three equations, is found the specific gravity of any body, as $s = \frac{\text{L}w}{\text{L} - l}$, by dividing the absolute weight of the body by its loss in water, and multiplying by the specific gravity of water.

But if the body L be lighter than water; then l will be negative, and we must divide by $\text{L} + l$ instead of $\text{L} - l$, and to find l we must have recourse to the compound mass C; and because, from the 4th and 5th equations, $\text{L} - l = \overline{\text{C} - c} - \overline{\text{H} - h}$, therefore $s = \frac{\text{L}w}{(\text{C} - c) - (\text{H} - h)}$; that is, divide the absolute weight of the light body, by the difference between the losses in water, of the compound and heavier body, and multiply by the specific gravity of water. Or thus, $s = \frac{\text{S}f\text{L}}{\text{CS} - \text{H}f}$, as found from the last equation.

Also, if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

H =

$$\text{H} = \frac{(f - s)\,\text{s}}{(\text{s} - s)f}\,\text{c}, \text{ and } \text{L} = \frac{(\text{s} - f)\,s}{(\text{s} - s)f}\,\text{c},$$

the quantities of the two ingredients H and L, in the compound C. And so for any other demand.

PROPOSITION LXV.

To find the Specific Gravity of a Body.

326. Case I.—*When the body is heavier than water:* weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then, by corol. 6, prop. 64, $s = \frac{\text{B}w}{\text{B} - b}$, where B is the weight of the body out of water, b its weight in water, s its specific gravity, and w the specific gravity of water. That is,

As the weight lost in water,
Is to the whole or absolute weight,
So is the specific gravity of water,
To the specific gravity of the body.

Example. If a piece of stone weigh 10 lb, but in water only $6\frac{3}{4}$ lb, required its specific gravity, that of water being 1000? Ans. 3077.

327. Case II.—*When the body is lighter than water, so that it will not sink:* annex to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass, separately, both in water, and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say, by proportion,

As the last remainder,
Is to the weight of the light body in air,
So is the specific gravity of water,
To the specific gravity of the body.

That is, the specific gravity is $s = \frac{\text{L}w}{(\text{C} - c) - (\text{H} - h)}$, by cor. 6, prop. 64.

Example. Suppose a piece of elm weighs 15 lb in air; and that a piece of copper, which weighs 18 lb in air and 16 lb in water, is affixed to it, and that the compound weighs 6 lb in water; required the specific gravity of the elm?

Ans. 600.

328. Case

328. CASE III.—*For a fluid of any sort.*—Take a piece of a body of known specific gravity; weigh it both in and out of the fluid, finding the loss of weight by taking the difference of the two; then say,

> As the whole or absolute weight,
> Is to the loss of weight,
> So is the specific gravity of the solid,
> To the specific gravity of the fluid.

That is, the spec. grav. $w = \frac{B - b}{B} s$, by cor. 6, pr. 64.

EXAMPLE. A piece of cast iron weighed $35\frac{61}{100}$ ounces in a fluid, and 40 ounces out of it; of what specific gravity is that fluid? Ans. 1000.

PROPOSITION LXVI.

329. *To find the Quantities of Two Ingredients in a Given Compound.*

TAKE the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply each specific gravity by the difference of the other two. Then say, by proportion,

> As the greatest product,
> Is to the whole weight of the compound,
> So is each of the other two products,
> To the weights of the two ingredients.

That is, $H = \frac{(f - s)\,S}{(S - s)f} C$ = the one, and $L = \frac{(S - f)\,s}{(S - s)f} C$, the other, by cor. 6, prop. 64.

EXAMPLE. A composition of 112 lb being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and that of copper 9000?

Answer, there is 100 lb of copper, and consequently 12 lb of tin, } in the composition.

SCHOLIUM.

330. The specific gravities of several sorts of matter, as found from experiments, are expressed by the numbers annexed to their names in the following Table:

A Table

A Table of Specific Gravities of Bodies.

Platina (pure) - -	23000	Clay - - - - -	2160
Fine gold - - -	19400	Brick - - - - -	2000
Standard gold - -	17724	Common earth - -	1984
Quicksilver (pure) -	14000	Nitre - - - - -	1900
Quicksilver (common)	13600	Ivory - - - - - -	1825
Lead - - - - -	11325	Brimstone - - -	1810
Fine silver - - -	11091	Solid gunpowder -	1745
Standard silver - -	10535	Sand - - - - - -	1520
Copper - - - - -	9000	Coal - - - - - -	1250
Copper halfpence -	8915	Box-wood - - -	1030
Gun metal - - - -	8784	Sea-water - - - -	1030
Cast brass - - -	8000	Common-water - -	1000
Steel - - - - - -	7850	Oak - - - - - -	925
Iron - - - - - -	7645	Gunpowder, close shaken	937
Cast iron - - - -	7425	Ditto, in a loose heap	836
Tin - - - - - - -	7320	Ash - - - - - -	800
Clear crystal glass -	3150	Maple - - - - - -	755
Granite - - - - -	3000	Elm - - - - - - -	600
Marble and hard stone	2700	Fir - - - - - -	550
Common green glass	2600	Charcoal - - - -	
Flint - - - - - -	2570	Cork - - - - - -	240
Common stone - -	2520	Air at a mean state -	$1\frac{2}{9}$

331. *Note.* The several sorts of wood are supposed to be dry. Also, as a cubic foot of water weighs just 1000 ounces avoirdupois, the numbers in this table express, not only the specific gravities of the several bodies, but also the weight of a cubic foot of each, in avoirdupois ounces; and therefore, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the next two propositions.

PROPOSITION LXVII.

332. *To find the Magnitude of any Body, from its Weight.*

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, respectively.

Example 1. Required the content of an irregular block of common stone, which weighs 1 cwt, or 112 lb?

Ans. $1228\frac{2016}{2520}$ cubic inches.

Example 2. How many cubic inches of gunpowder are there in 1 lb. weight? Ans. $29\frac{1}{2}$ cubic inches nearly.

Example 3.

Example 3. How many cubic feet are there in a ton weight of dry oak? Ans. $38\frac{138}{187}$ cubic feet.

PROPOSITION LXVIII.

333. *To find the Weight of a Body from its Magnitude.*

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is the tabular specific gravity,
To the weight of the body.

Example 1. Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Ans. $683\frac{4}{10}$ ton, which is nearly equal to the burden of an East-India ship.

Example 2. What is the weight of 1 pint, ale measure, of gunpowder? Ans. 19 oz. nearly.

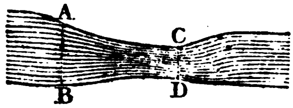
Example 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{4}$ feet deep or thick? Ans. $4335\frac{15}{16}$ lb.

Of HYDRAULICS.

334. HYDRAULICS is the science which treats of the motion of fluids, and the forces with which they act upon bodies.

PROPOSITION LXIX.

335. *If a Fluid Run through a Canal or River, or Pipe of various Widths, always filling it; the Velocity of the Fluid in different Parts of it* AB, CD, *will be reciprocally as the Transverse Sections in those Parts.*



THAT is, veloc. at A : veloc. at C :: CD : AB; where AB and CD denote, not the diameters at A and B, but the areas or sections there.

For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the same time; that is, the column through

AB is equal to the column through CD, in the same time; or AB × length of its column = CD × length of its column; therefore AB : CD :: length of column through CD : length of column through AB. But the uniform velocity of the water, is as the space run over, or length of the columns; therefore AB : CD :: velocity through CD : velocity through AB.

336. *Corol.* Hence, by observing the velocity at any place AB, the quantity of water discharged in a second, or any other time, will be found, namely, by multiplying the section AB by the velocity there.

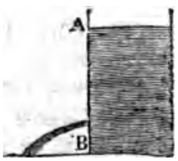
But if the channel be not a close pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the section will not be the same, because the velocity towards the bottom and sides will be diminished by the friction against the bed or channel; and therefore a medium among the three ought to be taken. So, if the velocity at the top be - 100 feet per minute,
that at the bottom - 60
and that at the sides - 50

3) 210 sum;

dividing their sum by 3, gives 70 for the mean velocity, which is to be multiplied by the section, to give the quantity discharged in a minute.

PROPOSITION LXX.

337. *The Velocity with which a Fluid Runs out by a Hole in the Bottom or Side of a Vessel, is Equal to that which is Generated by Gravity through the Height of the Water above the Hole; that is, the Velocity of a Heavy Body acquired by Falling freely through the Height* AB.



DIVIDE the altitude AB into a great number of very small parts, each being 1, their number a, or a = the altitude AB.

Now, by prop. 61, the pressure of the fluid against the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whose height is AB or a, and base the area of the hole B. Therefore the pressure on the hole, or small part of the fluid 1, is to its weight, or the natural force of gravity, as a to 1. But, by art. 28, the velocities generated in the same body in any time,

time, are as those forces; and because gravity generates the velocity 2 in descending through the small space 1, therefore $1 : a :: 2 : 2a$, the velocity generated by the pressure of the column of fluid in the same time. But $2a$ is also, by corol. 1 prop. 6, the velocity generated by gravity in descending through a or AB. That is, the velocity of the issuing water, is equal to that which is acquired by a body in falling through the height AB.

The same otherwise.

Because the momenta, or quantities of motion, generated in two given bodies, by the same force, acting during the same or an equal time, are equal. And as the force in this case, is the weight of the superincumbent column of the fluid over the hole. Let the one body to be moved, be that column itself, expressed by ah, where a denotes the altitude AB, and h the area of the hole; and the other body is the column of the fluid that runs out uniformly in one second suppose, with the middle or medium velocity of that interval of time, which is $\frac{1}{2}hv$, if v be the whole velocity required. Then the mass $\frac{1}{2}hv$, with the velocity v, gives the quantity of motion $\frac{1}{2}hv \times v$, or $\frac{1}{2}hv^2$, generated in one second, in the spouting water: also $2g$, or $32\frac{1}{6}$ feet, is the velocity generated in the mass ah, during the same interval of one second; consequently $ah \times 2g$, or $2ahg$, is the motion generated in the column ah in the same time of one second. But as these two momenta must be equal, this gives $\frac{1}{2}hv^2 = 2ahg$: hence then $v^2 = 4ag$, and $v = 2\sqrt{ag}$, for the value of the velocity sought; which therefore is exactly the same as the velocity generated by the gravity in falling through the space a, or the whole height of the fluid.

For example, if the fluid were air, of the whole height of the atmosphere, supposed uniform, which is about $5\frac{1}{4}$ miles, or 27720 feet $= a$. Then $2\sqrt{ag} = 2\sqrt{27720 \times 16\frac{1}{12}} =$ 1335 feet $= v$ the velocity, that is, the velocity with which common air would rush into a vacuum.

338. *Corol.* 1. The velocity, and quantity run out, at different depths, are as the square roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{AB}$.

339. *Corol.* 2. The fluid spouts out with the same velocity, whether it be downward or upward, or sideways; because the pressure of fluids is the same in all directions, at the same depth. And therefore, if an adjutage be turned upward, the jet will ascend to the height of the surface of the water in the vessel. And this is confirmed by experience, by which it is found that jets really ascend nearly to the height

height of the reservoir, abating a small quantity only, for the friction against the sides, and some resistance from the air and from the oblique motion of the fluid in the hole.

340. *Corol.* 3. The quantity run out in any time, is equal to a column or prism, whose base is the area of the hole, and its length the space described in that time by the velocity acquired by falling through the altitude of the fluid. And the quantity is the same, whatever be the figure of the orifice, if it is of the same area.

Therefore, if a denote the altitude of the fluid,
and b the area of the orifice,
also $g = 16\frac{1}{12}$ feet, or 193 inches;
then $2b\sqrt{ag}$ will be the quantity of water discharged in a second of time; or nearly $8\frac{1}{48}b\sqrt{a}$ cubic feet, when a and b are taken in feet.

So, for example, if the height a be 25 inches, and the orifice $b = 1$ square inch; then $2b\sqrt{ag} = 2\sqrt{25 \times 193} =$ 139 cubic inches, which is the quantity that would be discharged per second.

SCHOLIUM.

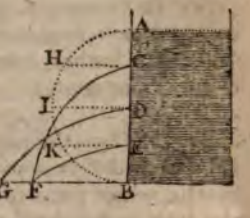
341. When the orifice is in the side of the vessel, then the velocity is different in the different parts of the hole, being less in the upper parts of it than in the lower. However, when the hole is but small, the difference is inconsiderable, and the altitude may be estimated from the centre of the whole, to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next proposition.

342. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the resistance of the air, the resistance of the water against the sides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles situated immediately in the column over the hole, which enter it and issue forth, as if that column only were in motion; but also particles from all the surrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, strike against each other, and impede one another's motion: from which it happens, that it is the particles in the centre of the hole only that issue out with the whole velocity due to the entire height of the fluid, while the other particles towards the sides of the orifices pass out with decreased velocities; and hence the medium velocity through the orifice, is somewhat less than that of a single body only, urged with the same pressure of the superincumbent column

of

of the fluid. And experiments on the quantity of water discharged through apertures, show that the quantity must be diminished, by those causes, rather more than the fourth part, when the orifice is small, or such as to make the mean velocity nearly equal to that in a body falling through $\frac{1}{2}$ the height of the fluid above the orifice.

343. Experiments have also been made on the extent to which the spout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile discharged with the velocity acquired by descending through the height of the fluid. For, when the aperture is in the side of the vessel, the fluid spouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, causes the jet to form the curve of a parabola. Then the distances to which the jet will spout on the horizontal plane BG, will be as the roots of the rectangles of the segments AC . CB, AD . DB, AE . EB. For the spaces BF, BG, are as the times and horizontal velocities; but the velocity is as $\sqrt{AC}$; and the time of the fall, which is the same as the time of moving, is as $\sqrt{CB}$; therefore the distance BF is as $\sqrt{AC . CB}$; and the distance BG as $\sqrt{AD . DB}$. And hence, if two holes are made equidistant from the top and bottom, they will project the water to the same distance; for if AC = EB, then the rectangle AC . CB is equal the rectangle AE . EB: which makes EF the same for both. Or, if on the diameter AB a semicircle be described; then, because the squares of the ordinates CH, DI, EK are equal to the rectangles AC . EB, &c; therefore the distances BF, BG are as the ordinates CH, DI. And hence also it follows, that the projection from the middle point D will be farthest, for DI is the greatest ordinate.



These are the proportions of the distances: but for the absolute distances, it will be thus. The velocity through any hole C, is such as will carry the water horizontally through a space equal to 2AC in the time of falling through AC: but, after quitting the hole, it describes a parabola, and comes to F in the time a body will fall through CB; and to find this distance, since the times are as the roots of the spaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC . CB} =$

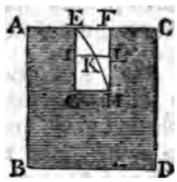
2CH

2CH = BF, the space ranged on the horizontal plane. And the greatest range BG = 2DI, or 2AD, or equal to AB.

And as these ranges answer very exactly to the experiments, this confirms the theory, as to the velocity assigned.

PROPOSITION LXXI.

344. *If a Notch or Slit* EH *in form of a Parallelogram, be cut in the Side of a Vessel, Full of Water,* AD; *the Quantity of Water flowing through it, will be* $\frac{2}{3}$ *of the Quantity flowing through an Equal Orifice, placed at the Whole Depth* EG, *or at the Base* GH, *in the Same Time; it being supposed that the Vessel is always kept full.*



For the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$; that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velocity at G, as the sum of all the ordinates IK, to the sum of all the IL's; namely, as the area of the parabola EGH, is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE, to EGHF, or as 2 to 3; the area of a parabola being $\frac{2}{3}$ of its circumscribing parallelogram.

345. *Corol.* 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

346. *Corol.* 2. The quantity flowing though the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. As appears from the demonstration.

Of PNEUMATICS.

347. Pneumatics is the science which treats of the properties of air, or elastic fluids.

PROPOSITION LXXII.

348. *Air is a Heavy Fluid Body; and it Surrounds the Earth, and Gravitates on all Parts of its Surface.*

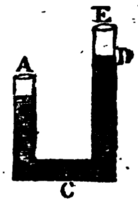
These properties of air are proved by experience.—That it is a fluid, is evident from its easily yielding to any

the least force impressed on it, without making a sensible resistance.

But when it is moved briskly, by any means, as by a fan or a pair of bellows; or when any body is moved very briskly through it; in these cases we become sensible of it as a body, by the resistance it makes in such motions; and also by its impelling or blowing away any light substances. So that, being capable of resisting, or moving other bodies, by its impulse, it must itself be a body, and be heavy, like all other bodies, in proportion to the matter it contains; and therefore it will press on all bodies that are placed under it.

Also, as it is a fluid, it spreads itself all over on the earth; and, like other fluids, it gravitates and presses everywhere on the earth's surface.

349. The gravity and pressure of the air is also evident from many experiments. Thus, for instance, if water, or quicksilver, be poured into the tube ACE, and the air be suffered to press on it, in both ends of the tube, the fluid will rest at the same height in both legs: but if the air be drawn out of one end as B, by any means; then the air pressing on the other end A, will press down the fluid in this leg at B, and raise it up in the other to D, as much higher than at B, as the pressure of the air is equal to. From which it appears, not only that the air does really press, but also how much the intensity of that pressure is equal to. And this is the principle of the barometer.



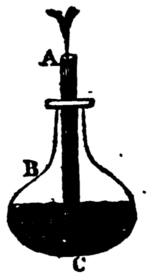
PROPOSITION LXXIII.

350. *The Air is also an Elastic Fluid, being Condensible and Expansible. And the Law it observes is this, that its Density and Elasticity are proportional to the Force or Weight which Compresses it.*

This property of the air is proved by many experiments. Thus, if the handle of a syringe be pushed inward, it will condense the inclosed air into less space, thereby showing its condensibility. But the included air, thus condensed, is felt to act strongly against the hand, resisting the force compressing it more and more; and, on withdrawing the hand, the handle is pushed back again to where it was at first. Which shows that the air is elastic.

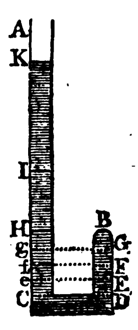
351. Again,

351. Again, fill a strong bottle half full of water; then insert a small glass tube into it, putting its lower end down near to the bottom, and cementing it very close round the mouth of the bottle. Then, if air be strongly injected through the pipe, as by blowing with the mouth or otherwise, it will pass through the water from the lower end, ascending into the parts before occupied with air at B, and the whole mass of air become there condensed, because the water is not compressible into a less space. But, on removing the force which injected the air at A, the water will begin to rise from thence in a jet, being pushed up the pipe by the increased elasticity of the air B, by which it presses on the surface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the same density as at first, and, the balance being restored, the jet will cease.



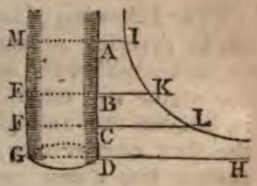
352. Likewise, if into a jar of water AB, be inverted an empty glass tumbler CD, or such-like, the mouth downward; the water will enter it, and partly fill it, but not near so high as the water in the jar, compressing and condensing the air into a less space in the upper parts C, and causing the glass to make a sensible resistance to the hand in pushing it down. Then, on removing the hand, the elasticity of the internal condensed air throws the glass up again. All these showing that the air is condensible and elastic.

353. Again, to show the rate or proportion of the elasticity to the condensation: take a long crooked glass tube, equally wide throughout, or at least in the part BD, and open at A; but close at the other end B. Pour in a little quicksilver at A, just to cover the bottom to the bend at CD, and to stop the communication between the external air and the air in BD. Then pour in more quicksilver, and mark the corresponding heights at which it stands in the two legs: so, when it rises to H in the open leg AC, let it rise to E in the close one, reducing its included air from the natural bulk BD to the contracted space BE;



by the pressure of the column He; and when the quicksilver stands at I and K, in the open leg, let it rise to F and G in the other, reducing the air to the respective spaces BF, BG, by the weights of the columns If, Kg. Then it is always found, that the condensations and elasticities are as the compressing weights or columns of the quicksilver, and the atmosphere together. So, if the natural bulk of the air BD be compressed into the spaces BE, BF, BG, which are $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$ of BD, or as the numbers 3, 2, 1; then the atmosphere, together with the corresponding columns He, If, Kg, are also found to be in the same proportion reciprocally, viz. as $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$, or as the numbers 2, 3, 6. And then $He = \frac{1}{3}A$, $If = A$, and $Kg = 3A$; where A is the weight of atmosphere. Which show, that the condensations are directly as the compressing forces. And the elasticities are in the same ratio, since the columns in AC are sustained by the elasticities in BD.

From the foregoing principles may be deduced many useful remarks, as in the following corollaries, viz.



354. *Corol.* 1. The space which any quantity of air is confined in, is reciprocally as the force that compresses it. So, the forces which confine a quantity of air in the cylindrical spaces AG, BG, CG, are reciprocally as the same, or reciprocally as the heights AD, BD, CD. And therefore if to the two perpendicular lines DA, DH, as asymptotes, the hyperbola IKL be described, and the ordinates AI, BK, CL be drawn; then the forces which confine the air in the spaces AG, BG, CG, will be directly as the corresponding ordinates AI, BK, CL, since these are reciprocally as the abscisses AD, BD, CD, by the nature of the hyperbola.

355. *Corol.* 2. All the air near the earth is in a state of compression, by the weight of the incumbent atmosphere.

356. *Corol.* 3. The air is denser near the earth, than in high places; or denser at the foot of a mountain, than at the top of it. And the higher above the earth, the less dense it is.

357. *Corol.* 4. The spring or elasticity of the air, is equal to the weight of the atmosphere above it; and they will produce the same effects: since they always sustain and balance each other.

358. *Corol.* 5.

358. *Corol.* 5. If the density of the air be increased, preserving the same heat or temperature, its spring or elasticity is also increased, and in the same proportion.

359. *Corol.* 6. By the pressure and gravity of the atmosphere, on the surface of fluids, the fluids are made to rise in any pipes or vessels, when the spring or pressure within is decreased or taken off.

PROPOSITION LXXIV.

360. *Heat Increases the Elasticity of the Air, and Cold Diminishes it. Or, Heat Expands, and Cold Condenses the Air.*

This property is also proved by experience.

361. Thus, tie a bladder very close with some air in it; and lay it before the fire: then as it warms, it will more and more distend the bladder, and at last burst it, if the heat be continued, and increased high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was on this principle that the first air-balloons were made by Montgolfier: for, by heating the air within them, by a fire beneath, the hot air distends them to a size which occupies a space in the atmosphere, whose weight of common air exceeds that of the balloon.

362. Also, if a cup or glass, with a little air in it, be inverted into a vessel of water; and the whole be heated over the fire, or otherwise; the air in the top will expand till it fill the glass, and expel the water out of it; and part of the air itself will follow, by continuing or increasing the heat.

Many other experiments, to the same effect, might be adduced, all proving the properties mentioned in the proposition.

SCHOLIUM.

363. So that, when the force of the elasticity of air is considered, regard must be had to its heat or temperature; the same quantity of air being more or less elastic, as its heat is more or less. And it has been found, by experiment, that the elasticity is increased by the 435th part, for each degree of heat, of which there are 180 between the freezing and boiling heat of water.

364. *N. B.* Water expands about the $\frac{3}{20000}$ part, with each degree of heat. (Sir Geo. Shuckburgh, Philos. Trans. 1777, p. 560, &c.)

Also, the

$$\text{Spec. grav. of}\ \left.\begin{array}{ll}\text{air} & 1{\cdot}201 \text{ or } 1\frac{1}{5} \\ \text{water} & 1000 \\ \text{mercury} & 13592\end{array}\right\}\ \begin{array}{l}\text{when the barom. is } 29{\cdot}5, \\ \text{and the therm. is } 55^{\circ} \\ \text{which are their mean heights} \\ \text{in this country.}\end{array}$$

$$\text{Or thus,}\ \left.\begin{array}{ll}\text{air} & 1{\cdot}222 \text{ or } 1\frac{2}{9} \\ \text{water} & 1000 \\ \text{mercury} & 13600\end{array}\right\}\ \begin{array}{l}\text{when the barom. is } 30, \\ \text{and thermometer } 55.\end{array}$$

PROPOSITION LXXV.

365. *The Weight or Pressure of the Atmosphere, on any Base at the Earth's Surface, is Equal to the Weight of a Column of Quicksilver, of the Same Base, and the Height of which is between* 28 *and* 31 *inches.*

THIS is proved by the barometer, an instrument which measures the pressure of the air, and which is described below. For, at some seasons, and in some places, the air sustains and balances a column of mercury, of about 28 inches: but at other times it balances a column of 29, or 30, or near 31 inches high; seldom in the extremes 28 or 31, but commonly about the means 29 or 30. A variation which depends partly on the different degrees of heat in the air near the surface of the earth, and partly on the commotions and changes in the atmosphere, from winds and other causes, by which it is accumulated in some places, and depressed in others, being thereby rendered denser and heavier, or rarer and lighter; which changes in its state are almost continually happening in any one place. But the medium state is commonly about $29\frac{1}{2}$ or 30 inches.

366. *Corol.* 1. Hence the pressure of the atmosphere on every square inch at the earth's surface, at a medium, is very near 15 pounds avoirdupois, or rather $14\frac{3}{4}$ pounds. For, a cubic foot of mercury weighing 13600 ounces nearly, an inch of it will weigh 7·866 or almost 8 ounces, or nearly half a pound, which is the weight of the atmosphere for every inch of the barometer on a base of a square inch; and therefore 30 inches, or the medium height, weighs very near $14\frac{3}{4}$ pounds.

367. *Corol.* 2. Hence also the weight or pressure of the atmosphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high. For, *water* and quicksilver are in weight nearly as 1 to 13·6; so

so that the atmosphere will balance a column of water 13·6 times as high as one of quicksilver; consequently

13·6 times 28 inches = 381 inches, or $31\frac{3}{4}$ feet,
13·6 times 29 inches = 394 inches, or $32\frac{5}{6}$ feet,
13·6 times 30 inches = 408 inches, or 34 feet,
13·6 times 31 inches = 422 inches, or $35\frac{1}{6}$ feet.

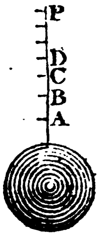
And hence a common sucking pump will not raise water higher than about 33 or 34 feet. And a siphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet.

368. *Corol.* 3. If the air were of the same uniform density at every height up to the top of the atmosphere, as at the surface of the earth; its height would be about $5\frac{1}{4}$ miles at a medium. For, the weights of the same bulk of air and water, are nearly as 1·222 to 1000; therefore as 1·222 : 1000 :: $33\frac{3}{4}$ feet: 27600 feet, or $5\frac{1}{4}$ miles nearly. And so high the atmosphere would be, if it were all of uniform density, like water. But, instead of that, from its expansive and elastic quality, it becomes continually more and more rare, the farther above the earth, in a certain proportion, which will be treated of below, as also the method of measuring heights by the barometer, which depends on it.

369. *Corol.* 4. From this proposition and the last it follows, that the height is always the same, of an uniform atmosphere above any place, which shall be all of the uniform density with the air there, and of equal weight or pressure with the real height of the atmosphere above that place, whether it be at the same place, at different times, or at any different places or heights above the earth; and that height is always about $5\frac{1}{4}$ miles, or 27600 feet, as above found. For, as the density varies in exact proportion to the weight of the column, therefore it requires a column of the same height in all cases, to make the respective weights or pressures. Thus, if W and w be the weights of atmosphere above any places, D and d their densities, and H and h the heights of the uniform columns, of the same densities and weights; Then $H \times D = W$, and $h \times d = w$; therefore $\frac{W}{D}$ or H is equal to $\frac{w}{d}$ or h. The temperature being the same.

PROPOSITION LXXVI.

370. *The Density of the Atmosphere, at Different Heights above the Earth, Decreases in such Sort, that when the Heights Increase in Arithmetical Progression, the Densities Decrease in Geometrical Progression.*



LET the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very small equal parts, A, B, C, D, &c, forming so many thin strata of air in the atmosphere, all of different density, gradually decreasing from the greatest at A: then the density of the several strata A, B, C, D, &c, will be in geometrical progression decreasing.

For, as the strata A, B, C, &c, are all of equal thickness, the quantity of matter in each of them, is as the density there; but the density in any one, being as the compressing force, is as the weight or quantity of all the matter from that place upward to the top of the atmosphere; therefore the quantity of matter in each stratum, is also as the whole quantity from that place upward. Now, if from the whole weight at any place as B, the weight or quantity in the stratum B be subtracted, the remainder is the weight at the next stratum C; that is, from each weight subtracting a part which is proportional to itself, leaves the next weight; or, which is the same thing, from each density subtracting a part which is proportional to itself, leaves the next density. But when any quantities are continually diminished by parts which are proportional to themselves, the remainders form a series of continued proportionals: consequently these densities are in geometrical progression.

Thus, if the first density be D, and from each be taken its nth part; there will then remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n-1$; and therefore the series of densities will be D, $\frac{m}{n}$D, $\frac{m^2}{n^2}$D, $\frac{m^3}{n^3}$D, $\frac{m^4}{n^4}$D, &c, the common ratio of the series being that of n to m.

SCHOLIUM.

371. Because the terms of an arithmetical series, are proportional to the logarithms of the terms of a geometrical series: therefore different altitudes above the earth's surface,

face, are as the logarithms of the densities, or of the weights of air, at those altitudes.

So that, if D denote the density at the altitude A,
and d - the density at the altitude a;
then A being as the log. of D, and a as the log. of d,

the dif. of alt. $A-a$ will be as the log. D − log. d. or log. $\frac{D}{d}$.

And if $A=0$, or D the density at the surface of the earth; then any altitude above the surface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the surface of the earth, or any where else.

And from this property is derived the method of determining the heights of mountains and other eminences, by the barometer, which is an instrument that measures the pressure or density of the air at any place. For, by taking, with this instrument, the pressure or density, at the foot of a hill for instance, and again at the top of it, the difference of the logarithms of these two pressures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; supposing the temperatures of the air to be the same at both places, and the gravity of air not altered by the different distances from the earth's centre.

372. But as this formula expresses only the relations between different altitudes with respect to their densities, recourse must be had to some experiment, to obtain the real altitude which corresponds to any given density, or the density which corresponds to a given altitude. And there are various experiments by which this may be done. The first, and most natural, is that which results from the known specific gravity of air, with respect to the whole pressure of the atmosphere on the surface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; assume h so that $a = h \times \log. \frac{D}{d}$, where h will be of one constant value for all altitudes; and to determine that value, let a case be taken in which we know the altitude a corresponding to a known density d; as for instance, take $a = 1$ foot, or 1 inch, or some such small altitude; then, because the density D may be measured by the pressure of the atmosphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will

denote the density D at the lower place, and 27599 the less density d at 1 foot above it; consequently $1 = h \times \log. \frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \frac{\cdot 43429448}{27600}$ $= \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log. \frac{D}{d}$, or $= 63551 \times \log. \frac{M}{m}$ feet, or $10592 \times \log. \frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the pressure or weight of the atmosphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any measure, either feet or inches, &c.

373. Note, that this formula is adapted to the mean temperature of the air 55°. But, for every degree of temperature different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which must be added, when that medium exceeds 55°, otherwise subtracted.

374. Note, also, that a column of 30 inches of mercury varies its length by about the $\frac{1}{320}$ part of an inch for every degree of heat, or rather $\frac{1}{9600}$ of the whole volume.

375. But the formula may be rendered much more convenient for use, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°; thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the corresponding change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the result is to be increased by so many times its 435th part.

376. *Exam.* 1. To find the height of a hill when the pressure of the atmosphere is equal to 29·68 inches of mercury at the bottom, and 25·28 at the top; the mean temperature being 50°? Ans. 4378 feet, or 730 fathoms.

377. *Exam.* 2. To find the height of a hill when the atmosphere weighs 29·45 inches of mercury at the bottom, and 26·82 at the top, the mean temperature being 33°?
Ans. 2385 feet, or 397½ fathoms.

378. *Exam.* 3.

378. *Exam.* 3. At what altitude is the density of the atmosphere only the 4th part of what it is at the earth's surface? Ans. 6020 fathoms.

By the weight and pressure of the atmosphere, the effect and operations of pneumatic engines may be accounted for, and explained; such as siphons, pumps, barometers, &c; of which it may not be improper here to give a brief description.

OF THE SIPHON.

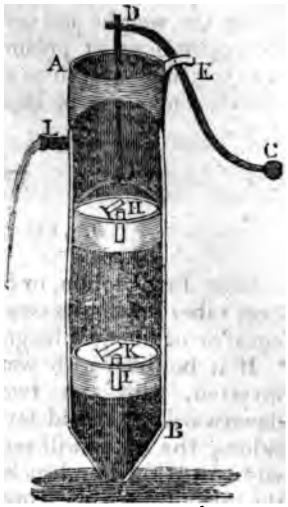
379. THE Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.

If it be filled with water, and then inverted, with the two open ends downward, and held level in that position; the water will remain suspended in it, if the two legs be equal. For the atmosphere will press equally on the surface of the water in each end, and support them, if they are not more than 34 feet high; and the legs being equal, the water in them is an exact counterpoise by their equal weights; so that the one has no power to move more than the other; and they are both supported by the atmosphere.

But if now the siphon be a little inclined to one side, so that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the same thing; then the equilibrium is destroyed, and the water will all descend out by the lower end, and rise up in the higher. For, the air pressing equally, but the two ends weighing unequally, a motion must commence where the power is greatest, and so continue till all the water has run out by the lower end. And if the shorter leg be immersed into a vessel of water, and the siphon be set a running as above, it will continue to run till all the water be exhausted out of the vessel, or at least as low as that end of the siphon. Or, it may be set a running without filling the siphon as above, by only inverting it, with its shorter leg into the vessel of water; then, with the mouth applied to the lower orifice A, suck the air out; and the water will presently follow, being forced up into the siphon by the pressure of the air on the water in the vessel.

OF

Of the PUMP.



380. There are three sorts of pumps: the Sucking, the Lifting, and the Forcing Pump. By the first, water can be raised only to about 34 feet; viz. by the pressure of the atmosphere; but by the others, to any height; but then they require more apparatus and power.

The annexed figure represents a common sucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and smooth within, or of wood for very common purposes. CD is the handle, moveable about the pin E, by moving the end C up and down. DF an iron rod turning about a pin D, which connects it to the end of the handle. This rod is fixed to the piston, bucket, or sucker, FG, by which this is moved up and down within the barrel, which it must fit very tight and close, that no air or water may pass between the piston and the sides of the barrel; and for this purpose it is commonly armed with leather. The piston is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel, also perforated, and covered by a valve K opening upwards.

381. When the pump is first to be worked, and the water is below the plug I; raise the end C of the handle, then the piston descending, compresses the air in HI, which by its spring shuts fast the valve K, and pushes up the valve H, and so enters into the barrel above the piston. Then putting the end C of the handle down again, raises the piston or sucker, which lifts up with it the column of air above it, the external atmosphere by its pressure keeping the valve H shut: the air in the barrel being thus exhausted, or rarefied, is no longer a counterpoise to that which presses on the surface of the water in the well; this is forced up the pipe, and through the valve K, into the barrel of the pump. Then pushing the piston down again into this water, now in the barrel,

barrel, its weight shuts the lower valve K, and its resistance forces up the valve of the piston, and enters the upper part of the barrel, above the piston. Then, the bucket being raised, lifts up with it the water which had passed above its valve, and it runs out by the cock L; and taking off the weight below it, the pressure of the external atmosphere on the water in the well again forces it up through the pipe and lower valve close to the piston, all the way as it ascends, thus keeping the barrel always full of water. And thus, by repeating the strokes of the piston, a continued discharge is made at the cock L.

OF THE AIR-PUMP.

382. NEARLY on the same principles as the water-pump, is the invention of the Air-pump, by which the air is drawn out of any vessel, like as water is drawn out by the former. A brass barrel is bored and polished truly cylindrical, and exactly fitted with a turned piston, so that no air can pass by the sides of it, and furnished with a proper valve opening upward. Then, by lifting up the piston, the air in the close vessel below it follows the piston, and fills the barrel; and being thus diffused through a larger space than before, when it occupied the vessel or receiver only, but not the barrel, it is made rarer than it was before, in proportion as the capacity of the barrel and receiver together exceeds the receiver alone. Another stroke of the piston exhausts another barrel of this now rarer air, which again rarifies it in the same proportion as before. And so on, for any number of strokes of the piston, still exhausting in the same geometrical progression, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhausted to any proposed degree, or as far as the nature of the machine is capable of performing; which happens when the elasticity of the included air is so far diminished, by rarefying, that it is too feeble to push up the valve of the piston, and escape.

383. From the nature of this exhausting, in geometrical progression, we may easily find how much the air in the receiver is rarefied by any number of strokes of the piston; or what number of such strokes is necessary, to exhaust the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone,

as

as c to r, and 1 denote the natural density of the air at first; then

$c : r :: 1 : \frac{r}{c}$, the density after 1 stroke of the piston,

$c : r :: \frac{r}{c} : \frac{r^2}{c^2}$, the density after 2 strokes,

$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}$, the density after three strokes,

&c, and $\frac{r^n}{c^n}$, the density after n strokes.

So, if the barrel be equal to $\frac{1}{4}$ of the receiver; then $c : r :: 5 : 4$; and $\frac{4^n}{5^n} = 0.8^n$ is $= d$ the density after n turns. And if n be 20, then $0{\cdot}8^{20} = {\cdot}0115$ is the density of the included air after 20 strokes of the piston; which being the $86\frac{7}{10}$ part of 1, or the first density, it follows that the air is $86\frac{7}{10}$ times rarefied by the 20 strokes.

384. Or, if it were required to find the number of strokes necessary to rarefy the air any number of times; because $\frac{r^n}{c^n}$ is = the proposed density d; therefore, taking the logarithms, $n \times \log.\frac{r}{c} = \log. d$, and $n = \frac{\log. d}{\text{l}. r - \text{l}. c}$, the number of strokes required. So if r be $\frac{4}{5}$ of c, and it be required to rarefy the air 100 times: then $d = \frac{1}{100}$ or ·01; and hence $n = \frac{\log. 100}{\text{l}. 5 - \text{l}. 4} = 20\frac{3}{5}$ nearly. So that in $20\frac{3}{5}$ strokes the air will be rarefied 100 times.

OF THE DIVING BELL & CONDENSING MACHINE.

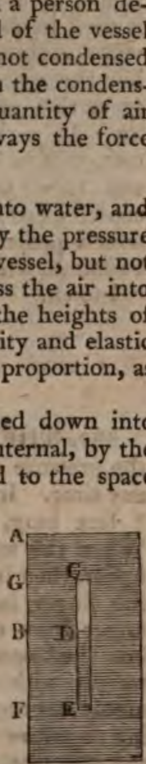
385. ON the same principles too depend the operations and effect of the Condensing Engine, by which air may be condensed to any degree, instead of rarefied as in the air-pump. And, like as the air-pump rarefies the air, by extracting always one barrel of air after another; so, by this other machine, the air is condensed, by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piston and barrel, that is, making them to open the contrary way, and working the piston in the same manner; so

so that, as they both open upward or outward in the air-pump, or rarefier, they will both open downward or inward in the condenser.

386. And on the same principles, namely, of the compression and elasticity of the air, depends the use of the Diving Bell, which is a large vessel, in which a person descends to the bottom of the sea, the open end of the vessel being downward; only in this case the air is not condensed by forcing more of it into the same space, as in the condensing engine; but by compressing the same quantity of air into a less space in the bell, by increasing always the force which compresses it.

387. If a vessel of any sort be inverted into water, and pushed or let down to any depth in it; then by the pressure of the water some of it will ascend into the vessel, but not so high as the water without, and will compress the air into less space, according to the difference between the heights of the internal and external water; and the density and elastic force of the air will be increased in the same proportion, as its space in the vessel is diminished.

So, if the tube CE be inverted, and pushed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the space CD; then that air is pressed both by a column of water of the height AB, and by the whole atmosphere which presses on the upper surface of the water; consequently the space CD is to the whole space CE, as the weight of the atmosphere, is to the weights both of the atmosphere and the column of water AB. So that, if AB be about 34 feet, which is equal to the force of the atmosphere, then CD will be equal to $\frac{1}{2}$CE; but if AB be double of that, or 68 feet, then CD will be $\frac{1}{3}$CE; and so on. And hence, by knowing the depth AF, to which the vessel is sunk, we can easily find the point D, to which the water will rise within it at any time. For let the weight of the atmosphere at that time be equal to that of 34 feet of water; also, let the depth AF be 20 feet, and the length of the tube CE 4 feet: then, putting the height of the internal water $DE = x$,



it is $34 + AB : 34 :: CE : CD$,

that is $34 + AF - DE : 34 :: CE : CE - DE$,

or $54 - x : 34 :: 4 : 4 - x$;

hence, multiplying extremes and means, $216 - 58x + x^2 = 136$,

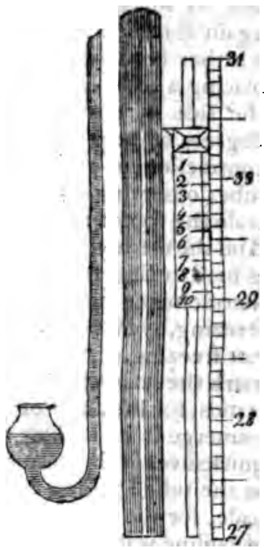
$= 136$, and the root is $x = \sqrt{2}$ very nearly $= 1{\cdot}414$ of a foot, or 17 inches nearly; being the height DE to which the water will rise within the tube.

388. But if the vessel be not equally wide throughout, but of any other shape, as of a bell-like form, such as is used in diving; then the altitudes will not observe the proportion above, but the spaces or bulks only will respect that proportion, namely, $34 +$ AB : 34 : : capacity CKL : capacity CHI, if it be common or fresh water; and $33 +$ AB : 33 : : capacity CKL : capacity CHI, if it be sea-water. From which proportion, the height DE may be found, when the nature or shape of the vessel or bell CKL is known.

Of the BAROMETER.

389. THE Barometer is an instrument for measuring the pressure of the atmosphere, and elasticity of the air, at any time. It is commonly made of a glass tube, of near 3 feet long, close at one end, and filled with mercury. When the tube is full, by stopping the open end with the finger, then inverting the tube, and immersing that end with the finger into a bason of quicksilver, on removing the finger from the orifice, the fluid in the tube will descend into the bason, till what remains in the tube be of the same weight with a column of the atmosphere, which is commonly between 28 and 31 inches of quicksilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no pressure downwards but from the column of quicksilver, and therefore that will be an exact balance to the counter pressure of the whole column of atmosphere, acting on the orifice of the tube by the quicksilver in the bason. The upper 3 inches of the tube, namely, from 28 to 31 inches, have a scale attached to them, divided into inches, tenths, and hundredths, for measuring the length of the column at all times, by observing which division of the scale the top of the quicksilver is opposite to; as it ascends and descends within these limits, according to the state of the atmosphere.

So



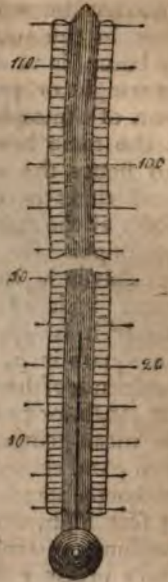
So that the weight of the quicksilver in the tube, above that in the bason, is at all times equal to the weight or pressure of the column of atmosphere above it, and of the same base with the tube; and hence the weight of it may at all times be computed; being nearly at the rate of half a pound avoirdupois for every inch of quicksilver in the tube, on every square inch of base; or more exactly it is $\frac{59}{120}$ of a pound on the square inch, for every inch in the altitude of the quicksilver weighs just $\frac{59}{120}$lb, or nearly $\frac{1}{2}$ a pound, in the mean temperature of 55° of heat. And consequently, when the barometer stands at 30 inches, or $2\frac{1}{2}$ feet high, which is nearly the medium or standard height, the whole pressure of the atmosphere is equal to $14\frac{3}{4}$ pounds, on every square inch of the base: and so in proportion for other heights.

Of the THERMOMETER.

390. THE Thermometer is an instrument for measuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold: and hence the degrees of expansion become the measure of the degrees of heat. Fluids are more convenient for this purpose than solids: and quicksilver is now most commonly used for it. A very fine glass tube, having a pretty large hollow ball at the bottom, is filled about half way up with quicksilver: the whole being then heated very hot till the quicksilver rise quite to the top, the top is then hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the quicksilver contracts, and consequently its surface descends in the tube, till it come to a certain point, correspondent to the temperature or heat of the air. And when the weather becomes warmer, the quicksilver expands,

and



and its surface rises in the tube; and again contracts and descends when the weather becomes cooler. So that, by placing a scale of any divisions against the side of the tube, it will show the degrees of heat by the expansion and contraction of the quicksilver in the tube; observing at what division of the scale the top of the quicksilver stands. And the method of preparing the scale, as used in England, is thus:—Bring the thermometer into the temperature of freezing, by immersing the ball in water just freezing, or in ice just thawing, and mark the scale where the mercury then stands, for the point of freezing. Next, immerge it in boiling water; and the quicksilver will rise to a certain height in the tube; which mark also on the scale, for the boiling point, or the heat of boiling water. Then the distance between these two points, is divided into 180 equal divisions, or degrees; and the like equal degrees are also continued to any extent below the freezing point, and above the boiling point. The divisions are then numbered as follows, namely, at the freezing point is set the number 32, and consequently 212 at the boiling point; and all the other numbers in their order.

This division of the scale is commonly called Fahrenheit's. According to this division, 55 is at the mean temperature of the air in this country; and it is in this temperature, and in an atmosphere which sustains a column of 30 inches of quicksilver in the barometer, that all measures and specific gravities are taken, unless when otherwise mentioned; and in this temperature and pressure, the relative weights, or specific gravities of air, water, and quicksilver, are as

$1\frac{2}{9}$ for air, 1000 for water, 13600 for mercury; { and these also are the weights of a cubic foot of each, in avoirdupois ounces, in that state of the barometer and thermometer. For other states of the thermometer, each of these bodies expands or contracts according to the following rate, with each degree of heat, viz.

Air about - $\frac{1}{435}$ part of its bulk,
Water about $\frac{1}{6666}$ part of its bulk,
Mercury about $\frac{1}{9600}$ part of its bulk.

ON

ON THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

391. FROM the principles laid down in the scholium to prop. 76, concerning the measuring of altitudes by the barometer, and the foregoing descriptions of the barometer and thermometer, we may now collect together the precepts for the practice of such measurements, which are as follow:

First. Observe the height of the barometer at the bottom of any height, or depth, intended to be measured; with the temperature of the quicksilver, by means of a thermometer attached to the barometer, and also the temperature of the air in the shade by a detached thermometer.

Secondly. Let the same thing be done also at the top of the said height or depth, and at the same time, or as near the same time as may be. And let those altitudes of barometer be reduced to the same temperature, if it be thought necessary, by correcting either the one or the other, that is, augment the height of the mercury in the colder temperature, or diminish that in the warmer, by its $\frac{1}{9600}$ part for every degree of difference of the two.

Thirdly. Take the difference of the common logarithms of the two heights of the barometer, corrected as above if necessary, cutting off 3 figures next the right hand for decimals, when the log-tables go to 7 figures, or cut off only 2 figures when the tables go to 6 places, and so on; or in general remove the decimal point 4 places more towards the right hand, those on the left hand being fathoms in whole numbers.

Fourthly. Correct the number last found for the difference of temperature of the air, as follows; Take half the sum of the two temperatures, for the mean one; and for every degree which this differs from the temperature 31°, take so many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but subtract them if the mean temperature be below 31°; and the sum or difference will be the true altitude in fathoms: or, being multiplied by 6, it will be the altitude in feet.

392. *Example* 1. Let the state of the barometers and thermometers be as follows; to find the altitude, viz.

Barom.	Thermom. attach.	Thermom. detach.
Lower 29·68	57	57
Upper 25·28	43	42

Ans. the alt. is $719\frac{1}{3}$ fathoms.

393. *Exam.* 2. To find the altitude, when the state of the barometers and thermometers is as follows, viz.

Barom.	Thermom. attach.	Thermom. detach.	Ans. the alt. is
Lower 29·45	38	31	$409\frac{4}{13}$ fathoms,
Upper 26·82	41	35	or 2458 feet.

ON THE RESISTANCE OF FLUIDS, WITH THEIR FORCES AND ACTIONS ON BODIES.

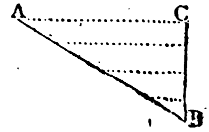
PROPOSITION LXXVII.

394. *If any Body Move through a Fluid at Rest, or the Fluid Move against the Body at Rest; the Force or Resistance of the Fluid against the Body, will be as the Square of the Velocity and the Density of the Fluid. That is,* $R \propto dv^2$.

FOR, the force or resistance is as the quantity of matter or particles struck, and the velocity with which they are struck. But the quantity or number of particles struck in any time, are as the velocity and the density of the fluid. Therefore the resistance, or force of the fluid, is as the density and square of the velocity.

395. *Corol.* 1. The resistance to any plane, is also more or less, as the plane is greater or less; and therefore the resistance on any plane, is as the area of the plane a, the density of the medium, and the square of the velocity. That is, $R \propto adv^2$.

396. *Corol.* 2. If the motion be not perpendicular, but oblique to the plane, or to the face of the body; then the resistance, in the direction of motion, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination of the plane to the direction of the motion, or as the cube of radius to the cube of the sine of that angle. So that $R \propto adv^2s^3$, putting $1 =$ radius, and $s =$ sine of the angle of inclination CAB.



For, if AB be the plane, AC the direction of motion, and BC perpendicular to AC; then no more particles meet the plane than what meet the perpendicular BC, and therefore their number is diminished as AB to BC or as 1 to s. But the force of each particle,

ticle, striking the plane obliquely in the direction CA, is also diminished as AB to BC, or as 1 to s; therefore the resistance, which is perpendicular to the face of the plane by art. 52, is as 1^2 to s^2. But again, this resistance in the direction perpendicular to the face of the plane, is to that in the direction AC, by art. 51, as AB to BC, or as 1 to s. Consequently, on all these accounts, the resistance to the plane when moving perpendicular to its face, is to that when moving obliquely, as 1^3 to s^3, or 1 to s^3. That is, the resistance in the direction of the motion, is diminished as 1 to s^3, or in the triplicate ratio of radius to the sine of inclination.

PROPOSITION LXXVIII.

397. *The Real Resistance to a Plane, by a Fluid acting in a Direction perpendicular to its Face, is equal to the Weight of a Column of the Fluid, whose Base is the Plane, and Altitude equal to that which is due to the Velocity of the Motion, or through which a Heavy Body must fall to acquire that Velocity.*

THE resistance to the plane moving through a fluid, is the same as the force of the fluid in motion with the same velocity, on the plane at rest. But the force of the fluid in motion, is equal to the weight or pressure which generates that motion; and this is equal to the weight or pressure of a column of the fluid, whose base is the area of the plane, and its altitude that which is due to the velocity.

398. *Corol.* 1. If a denote the area of the plane, v the velocity, n the density or specific gravity of the fluid, and $g = 16\frac{1}{12}$ feet, or 193 inches. Then, the altitude due to the velocity v being $\frac{v^2}{4g}$, therefore $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$ will be the whole resistance, or motive force R.

399. *Corol.* 2. If the direction of motion be not perpendicular to the face of the plane, but oblique to it, in any angle, whose sine is s. Then the resistance to the plane will be $\frac{anv^2s^3}{4g}$.

400. *Corol.* 3. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force R; then the retarding force f, or $\frac{R}{w}$ will be $\frac{anv^2s^3}{4gw}$.

401. *Corol.* 4. And if the body be a cylinder, whose face

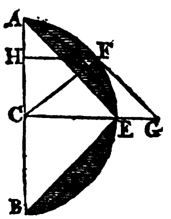
 or

or end is a, and radius r, moving in the direction of its axis; because then $s = 1$, and $a = pr^2$, where $p = 3{\cdot}1416$; then $\frac{pnv^2r^2}{4g}$ will be the resisting force R, and $\frac{pnv^2r^2}{4gw}$ the retarding force f.

402. *Corol.* 5. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face an elliptic section, or a conical surface, or any other figure everywhere equally inclined to the axis, or direction of motion, the sine or inclination being s: then, the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force R would be $\frac{pnr^2v^2s^2}{4g}$.

PROPOSITION LXXIX.

403. *The Resistance to a Sphere moving through a Fluid, is but Half the Resistance to its Great Circle, or to the End of a Cylinder of the same Diameter, moving with an Equal Velocity.*



LET AFEB be half the sphere, moving in the direction CEG. Describe the paraboloid AIEKB on the same base. Let any particle of the medium meet the semicircle in F, to which draw the tangent FG, the radius FC, and the ordinate FIH. Then the force of any particle on the surface at F, is to its force on the base at H, as the square of the sine of the angle G, or its equal the angle FCH, to the square of radius, that is, as HF^2 to CF^2. Therefore the force of all the particles, or the whole fluid, on the whole surface, is to its force on the circle of the base, as all the HF^2 to as many times CF^2. But CF^2 is $= CA^2 = AC \,.\, CB$, and $HF^2 = AH \,.\, HB$ by the nature of the circle: also, $AH \,.\, HB : AC \,.\, CB :: HI : CE$ by the nature of the parabola; consequently the force on the spherical surface, is to the force on its circular base, as all the HI's to as many CE's, that is, as the content of the paraboloid to the content of its circumscribed cylinder, namely, as 1 to 2.

404. *Corol.* Hence, the resistance to the sphere is $R = \frac{pnv^2r^2}{8g}$, being the half of that of a cylinder of the same diameter,

diameter. For example, a 9lb iron ball, whose diameter is 4 inches, when moving through the air with a velocity of 1600 feet per second, would meet a resistance which is equal to a weight of $132\frac{2}{3}$lb, over and above the pressure of the atmosphere, for want of the counterpoise behind the ball.

PRACTICAL EXERCISES IN MENSURATION.

QUEST. 1. WHAT difference is there between a floor 28 feet long by 20 broad, and two others, each of half the dimensions; and what do all three come to at 45s. per square, or 100 square feet?

Ans. dif. 280 sq. feet. Amount 18 guineas.

QUEST. 2. An elm plank is 14 feet 3 inches long, and I would have just a square yard slit off it; at what distance from the edge must the line be struck? Ans. $7\frac{11}{19}$ inches.

QUEST. 3. A cieling contains 114 yards 6 feet of plastering, and the room 28 feet broad; what is the length of it?

Ans. $36\frac{6}{7}$ feet.

QUEST. 4. A common joist is 7 inches deep, and $2\frac{1}{2}$ thick; but wanting a scantling just as big again, that shall be 3 inches thick; what will the other dimension be?

Ans. $11\frac{2}{3}$ inches.

QUEST. 5. A wooden cistern cost me 3*s.* 2*d.* painting within, at 6d. per yard; the length of it was 102 inches, and the depth 21 inches; what was the width?

Ans. $27\frac{1}{4}$ inches.

QUEST. 6. If my court-yard be 47 feet 9 inches square, and I have laid a foot-path with Purbeck stone, of 4 feet wide, along one side of it; what will paving the rest with flints come to, at 6d. per square yard? Ans. 5*l.* 16*s.* $0\frac{1}{2}$*d.*

QUEST. 7. A ladder, $26\frac{2}{3}$ feet long, may be so planted, that it shall reach a window 22 feet from the ground on one side of the street; and, by only turning it over, without moving the foot out of its place, it will do the same by a window 14 feet high on the other side; what is the breadth of the street? Ans. 37 feet $9\frac{1}{2}$ inches.

QUEST. 8. The paving of a triangular court, at 18d. per foot, came to 100*l.*; the longest of the three sides was 88 feet; required the sum of the other two equal sides?

Ans. 106·85 feet.

QUEST. 9. There are two columns in the ruins of Persepolis left standing upright: the one is 64 feet above the plain, and the other 50: in a straight line between these stands an ancient small statue, the head of which is 97 feet from the summit of the higher, and 86 feet from the top of the lower column, the base of which measures just 76 feet to the centre of the figure's base. Required the distance between the tops of the two columns? Ans. 157 feet nearly.

QUEST. 10. The perambulator, or surveying wheel, is so contrived, as to turn just twice in the length of 1 pole, or $16\frac{1}{2}$ feet; required the diameter? Ans. 2·626 feet.

QUEST. 11. In turning a one-horse chaise within a ring of a certain diameter, it was observed that the outer wheel made two turns, while the inner made but one: the wheels were both 4 feet high; and supposing them fixed at the distance of 5 feet asunder on the axletree, what was the circumference of the track described by the outer wheel?

Ans. 62·83 feet.

QUEST. 12. What is the side of that equilateral triangle, whose area cost as much paving at 8d. a foot, as the pallisading the three sides did at a guinea a yard?

Ans. 72·746 feet.

QUEST. 13. In the trapezium ABCD, are given, AB = 13, BC = $31\frac{1}{3}$, CD = 24, and DA = 18, also B a right angle; required the area? Ans. 410·122.

QUEST. 14. A roof which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead at 8lb. per square foot: what will it come to at 18s per cwt.? Ans. 22*l.* 19*s.* $10\frac{1}{4}$*d.*

QUEST. 15. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left: what will be the dimensions of the remaining piece? Ans. 20·7 inches by 6·086.

QUEST. 16. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land?

Ans. 58·876 or 23·099.

QUEST. 17. The end wall of a house is 24 feet 6 inches in breadth, and 40 feet to the eaves; $\frac{1}{3}$ of which is 2 bricks thick, $\frac{1}{3}$ more is $1\frac{1}{2}$ brick thick, and the rest 1 brick thick. Now the triangular gable rises 38 courses of bricks, 4 of which usually make a foot in depth, and this is but $4\frac{1}{2}$ inches,

or

or half a brick thick: what will this piece of work come to at 5*l*. 10*s*. per statute rod? Ans. 20*l*. 11*s*. 7¼*d*.

QUEST. 18. How many bricks will it take to build a wall, 10 feet high, and 500 feet long, of a brick and half thick; reckoning the brick 10 inches long, and 4 courses to the foot in height? Ans. 72000.

QUEST. 19. How many bricks will build a square pyramid of 100 feet on each side at the base, and also 100 feet perpendicular height: the dimensions of a brick being supposed 10 inches long, 5 inches broad, and 3 inches thick? Ans. 3840000.

QUEST. 20. If, from a right-angled triangle, whose base is 12, and perpendicular 16 feet, a line be drawn parallel to the perpendicular, cutting off a triangle whose area is 24 square feet; required the sides of this triangle? Ans. 6, 8, and 10.

QUEST. 21. The ellipse in Grosvenor-square measures 840 links across the longest way, and 612 the shortest, within the rails: now the walls being 14 inches thick, what ground do they enclose, and what do they stand upon?

Ans. $\begin{cases}\text{enclose 4 ac. 0 r. 6 p.}\\ \text{stand on } 1760\frac{1}{2} \text{ sq. feet.}\end{cases}$

QUEST. 22. If a round pillar, 7 inches over, have 4 feet of stone in it: of what diameter is the column, of equal length, that contains 10 times as much? Ans. 22·136 inches.

QUEST. 23. A circular fish-pond is to be made in a garden, that shall take up just half an acre; what must be the length of the chord that strikes the circle? Ans. 27¾ yards.

QUEST. 24. When a roof is of a true pitch, or making a right angle at the ridge, the rafters are nearly ¾ of the breadth of the building: now supposing the eaves-boards to project 10 inches on a side, what will the new ripping a house cost, that measures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15*s*. per square? Ans. 8*l*. 15*s*. 9½*d*.

QUEST. 25. A cable, which is 3 feet long, and 9 inches in compass, weighs 22lb; what will a fathom of that cable weigh, which measures a foot about? Ans. 78$\frac{2}{9}$ lb.

QUEST. 26. My plumber has put 28lb. per square foot into a cistern, 74 inches and twice the thickness of the lead long, 26 inches broad, and 40 deep: he has also put three stays across it within, of the same strength, and 16 inches deep,

deep, and reckons 22*s.* per cwt. for work and materials. I, being a mason, have paved him a workshop, 22 feet 10 inches broad, with Purbeck stone, at 7d. per foot; and on the balance I find there is 3*s.* 6*d.* due to him; what was the length of the workshop, supposing sheet lead of $\frac{1}{10}$ of an inch thick to weigh 5·899lb. the square foot?

Ans. 32 feet, $0\frac{3}{4}$ inch.

Quest. 27. The distance of the centres of two circles, whose diameters are each 50, being given, equal to 30; what is the area of the space enclosed by their circumferences?

Ans. 559·119.

Quest. 28. If 20 feet of iron railing weigh half a ton, when the bars are an inch and quarter square; what will 50 feet come to at $3\frac{1}{2}d.$ per lb, the bars being $\frac{7}{8}$ of an inch square? Ans. 20*l.* 0*s.* 2*d.*

Quest. 29. The area of an equilateral triangle, whose base falls on the diameter, and its vertex in the middle of the arc of a semicircle, is equal to 100: what is the diameter of the semicircle? Ans. 26·32148.

Quest. 30. It is required to find the thickness of the lead in a pipe, of an inch and quarter bore, which weighs 14lb. per yard in length; the cubic foot of lead weighing 11325 ounces? Ans. ·20737 inches.

Quest. 31. Supposing the expense of paving a semicircular plot, at 2*s.* 4*d.* per foot, come to 10*l.*; what is the diameter of it? Ans. 14·7737 feet.

Quest. 32. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle whose diameter is 289?

Ans. 278·6716.

Quest. 33. My plumber has set me up a cistern, and, his shop-book being burnt, he has no means of bringing in the charge, and I do not choose to take it down to have it weighed; but by measure he finds it contains $64\frac{3}{16}$ square feet, and that it is precisely $\frac{1}{8}$ of an inch in thickness. Lead was then wrought at 21*l.* per fother of $19\frac{1}{2}$ cwt. It is required from these items to make out the bill, allowing $6\frac{5}{9}$ oz. for the weight of a cubic inch of lead? Ans. 4*l.* 11*s.* 2*d.*

Quest. 34. What will the diameter of a globe be, when the solidity and superficial content are expressed by the same number? Ans. 6.

Quest. 35. A sack, that would hold 3 bushels of corn, is $22\frac{1}{2}$ inches broad when empty; what will another sack contain,

contain, which, being of the same length, has twice its breadth, or circumference? Ans. 12 bushels.

QUEST. 36. A carpenter is to put an oaken curb to a round well, at 8*d*. per foot square: the breadth of the curb is to be $7\frac{1}{4}$ inches, and the diameter within $3\frac{1}{2}$ feet; what will be the expense? Ans. 5*s*. $2\frac{1}{4}$*d*.

QUEST. 37. A gentleman has a garden 100 feet long, and 80 feet broad; and a gravel walk is to be made of an equal width half round it: what must the breadth of the walk be, to take up just half the ground? Ans. 25·968 feet.

QUEST. 38. The top of a may-pole, being broken off by a blast of wind, struck the ground at 10 feet distance from the foot of the pole; what was the height of the whole may-pole, supposing the length of the broken piece to be 26 feet? Ans. 50 feet.

QUEST. 39. Seven men bought a grinding stone, of 60 inches diameter, each paying $\frac{1}{7}$ part of the expense; what part of the diameter must each grind down for his share?
Ans. the 1st 4·4508, 2d 4·8400, 3d 5·3535, 4th 6·0765, 5th 7·2079, 6th 9·3935, 7th 22·6778 inches.

QUEST. 40. A maltster has a kiln, that is 16 feet 6 inches square: but he wants to pull it down, and build a new one, that may dry three times as much at once as the old one; what must be the length of its side? Ans. 28 feet 7 inches.

QUEST. 41. How many 3-inch cubes may be cut out of a 12-inch cube? Ans. 64.

QUEST. 42. How long must the tether of a horse be, that will allow him to graze, quite around, just an acre of ground? Ans. $39\frac{1}{4}$ yards.

QUEST. 43. What will the painting of a conical spire come to, at 8*d*. per yard; supposing the height to be 118 feet, and the circumference of the base 64 feet? Ans. 14*l*. 0*s*. $8\frac{3}{4}$*d*.

QUEST. 44. The diameter of a standard corn bushel is $18\frac{1}{2}$ inches, and its depth 8 inches; then what must the diameter of that bushel be, whose depth is $7\frac{1}{2}$ inches? Ans. 19·1067 inches.

QUEST. 45. Suppose the ball on the top of St. Paul's church is 6 feet in diameter; what did the gilding of it cost at $3\frac{1}{2}$ per square inch? Ans. 237*l*. 10*s*. 1*d*.

QUEST. 46. What will a frustum of a marble cone come to, at 12*s*. per solid foot; the diameter of the greater end being 4 feet, that of the less end $1\frac{1}{2}$, and the length of the slant side 8 feet? Ans. 30*l*. 1*s*. $10\frac{1}{4}$*d*.

QUEST. 47.

QUEST. 47. To divide a cone into three equal parts by sections parallel to the base, and to find the altitudes of the three parts, the height of the whole cone being 20 inches?

Ans. the upper part 13·867.
the middle part 3·605.
the lower part 2·528.

QUEST. 48. A gentleman has a bowling green, 300 feet long, and 200 feet broad, which he would raise 1 foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth must the ditch be dug, supposing its breadth to be every where 8 feet? Ans. $7\frac{23}{86}$ feet.

QUEST. 49. How high above the earth must a person be raised, that he may see $\frac{1}{3}$ of its surface?

Ans. to the height of the earth's diameter.

QUEST. 50. A cubic foot of brass is to be drawn into wire, of $\frac{1}{40}$ of an inch in diameter; what will the length of the wire be, allowing no loss in the metal?

Ans. 97784·797 yards, or 55 miles 984·797 yards.

QUEST. 51. Of what diameter must the bore of a cannon be, which is cast for a ball of 24lb. weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? Ans. 5·647 inches.

QUEST. 52. Supposing the diameter of an iron 9lb. ball to be 4 inches, as it is very nearly; it is required to find the diameters of the several balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 32, 36, and 42lb, and the caliber of their guns, allowing $\frac{1}{50}$ of the caliber, or $\frac{1}{49}$ of the ball's diameter, for windage.

Answer

Wt. of ball.	Diameter of ball.	Caliber of gun.
1	1·9230	1·9622
2	2·4228	2·4723
3	2·7734	2·8301
4	3·0526	3·1149
6	3·4943	3·5656
9	4·0000	4·0816
12	4·4026	4·4924
18	5·0397	5·1425
24	5·5469	5·6601
32	6·1051	6·2297
36	6·3496	6·4792
42	6·6844	6·8208

QUEST. 53.

QUEST. 53. Supposing the windage of all mortars to be $\frac{1}{60}$ of the caliber, and the diameter of the hollow part of the shell to be $\frac{7}{10}$ of the caliber of the mortar: it is required to determine the diameter and weight of the shell, and the quantity or weight of powder requisite to fill it, for each of the several sorts of mortars, namely, the 13, 10, 8, 5·8, and 4·6 inch mortar. Answer

Calib. of mort.	Diameter of shell.	Wt. of shell empty.	Wt. of powder.	Wt. of shell filled.
4·6	4·523	8·320	0·583	8·903
5·8	5·703	16·677	1·168	17·845
8	7·867	43·764	3·065	46·829
10	9·833	85·476	5·986	91·462
13	12·783	187·791	13·151	200·942

QUEST. 54. If a heavy sphere, whose diameter is 4 inches, be let fall into a conical glass, full of water, whose diameter is 5, and altitude 6 inches; it is required to determine how much water will run over?

Ans. 26·272 cubic inches, or nearly $\frac{3}{4}$ of a pint.

QUEST. 55. The dimensions of the sphere and cone being the same as in the last question, and the cone only $\frac{1}{5}$ full of water; required what part of the axis of the sphere is immersed in the water? Ans. ·546 parts of an inch.

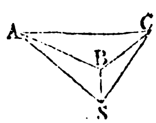
QUEST. 56. The cone being still the same, and $\frac{1}{5}$ full of water; required the diameter of a sphere which shall be just all covered by the water? Ans. 2·445996 inches.

QUEST. 57. If a person, with an air balloon, ascend vertically from London, to such a height that he can just see Oxford appear in the horizon; it is required to determine his height above the earth, supposing its circumference to be 25000 miles, and the distance between London and Oxford 49·5933 miles? Ans. $\frac{311}{1000}$ of a mile, or 547 yards 1 foot.

QUEST. 58. In a garrison there are three remarkable objects A, B, C, the distances of which from one to another are known to be, AB 213, AC 424, and BC 262 yards; I am desirous of knowing my position and distance at a place or station S, from which I observed the angle ASB 13° 30′, and the angle CSB 29° 50′, both by geometry and trigonometry.

Answer,

AS 605·7122;
BS 429·6814;
CS 524·2365.

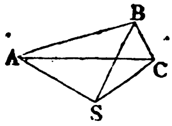


QUEST.

QUEST. 59. Required the same as in the last question, when the point B is on the other side of AC, supposing AB 9, AC 12, and BC 6 furlongs; also the angle ASB 33° 45′, and the angle BSC 22° 30′.

Answer,

AS 10·64, BS 15·64, CS 14·01.



QUEST. 60. It is required to determine the magnitude of a cube of gold, of the standard fineness, which shall be equal to a sum of 480 million of pounds sterling; supposing a guinea to weigh 5 dwts $9\frac{1}{4}$ grains. Ans. 18·691 feet.

QUEST. 61. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch?

Ans. 1158127 gallons nearly.

QUEST. 62. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles: required the ratio of their surfaces, and also of their solidities: supposing them both to be globular, as they are very nearly?

Ans. the surfaces are as $13\frac{1}{2}$ to 1 nearly; and the solidities as $49\frac{1}{2}$ to 1 nearly.

PRACTICAL EXERCISES CONCERNING SPECIFIC GRAVITY.

The Specific Gravities of Bodies are their relative weights contained under the same given magnitude; as a cubic foot, or a cubic inch, &c.

The specific gravities of several sorts of matter, are expressed by the numbers annexed to their names in the Table of Specific Gravities, at page 231; from which the numbers are to be taken, when wanted.

Note. The several sorts of wood are supposed to be dry. Also, as a cubic foot of water weighs just 1000 ounces avoirdupois, the numbers in the table express, not only the specific gravities of the several bodies, but also the weight of a cubic foot of each in avoirdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity

quantity of any other weight, may be known, as in the following problems.

PROBLEM I.

To find the Magnitude of any Body, from its Weight.

As the tabular specific gravity of the body,
Is to its weight in avoirdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, respectively.

EXAMPLES.

EXAM. 1. Required the content of an irregular block of common stone, which weighs 1cwt. or 112lb.

Ans. $1228\frac{4}{5}$ cubic inches.

EXAM. 2. How many cubic inches of gunpowder are there in 1lb weight? Ans. $29\frac{1}{2}$ cubic inches nearly.

EXAM. 3. How many cubic feet are there in a ton weight of dry oak? Ans. $38\frac{138}{185}$ cubic feet.

PROBLEM II.

To find the Weight of a Body from its Magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular specific gravity,
To the weight of the body.

EXAMPLES.

EXAM. 1. Required the weight of a block of marble, whose length is 63 feet, and breadth and thickness each 12 feet; being the dimensions of one of the stones in the walls of Balbeck?

Ans. $683\frac{7}{16}$ ton, which is nearly equal to the burden of an East-India ship.

EXAM. 2. What is the weight of 1 pint, ale measure, of gunpowder? Ans. 19 oz. nearly.

EXAM. 3. What is the weight of a block of dry oak, which measures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep? Ans. $4335\frac{15}{16}$lb.

PROBLEM

PROBLEM III.

To find the Specific Gravity of a Body.

CASE 1. When the body is heavier than water, weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then say,

As the weight lost in water,
Is to the whole weight,
So is the specific gravity of water,
To the specific gravity of the body.

EXAMPLE.

A piece of stone weighed 10lb, but in water only 6¼lb, required its specific gravity? Ans. 2609.

CASE 2. When the body is lighter than water, so that it will not quite sink, affix to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound mass separately, both in water and out of it; then find how much each loses in water, by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then say,

As the last remainder,
Is to the weight of the light body in air,
So is the specific gravity of water,
To the specific gravity of the body.

EXAMPLE.

Suppose a piece of elm weighs 15lb in air; and that a piece of copper which weighs 18lb in air, and 16lb in water, is affixed to it, and that the compound weighs 6lb in water; required the specific gravity of the elm? Ans. 600.

PROBLEM IV.

To find the Quantities of Two Ingredients in a Given Compound.

TAKE the three differences of every pair of the three specific gravities, namely, the specific gravities of the compound and each ingredient; and multiply the difference of every two specific gravities by the third. Then say, as the greatest product, is to the whole weight of the compound, so is each of the other products, to the two weights of the ingredients.

EXAMPLE.

EXAMPLE.

A composition of 112lb being made of tin and copper, whose specific gravity is found to be 8784; required the quantity of each ingredient, the specific gravity of tin being 7320, and of copper 9000?

Ans. there is 100lb of copper and consequently 12lb of tin } in the composition.

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimensions of Balls and Shells might be found from the problems last given, concerning specific gravity. But they may be found still easier by means of the experimented weight of a ball of a given size, from the known proportion of similar figures, namely, as the cubes of their diameters.

PROBLEM I.

To find the Weight of an Iron Ball, from its Diameter.

An iron ball of 4 inches diameter weighs 9lb, and the weights being as the cubes of the diameters, it will be, as 64 (which is the cube of 4) is to 9 its weight, so is the cube of the diameter of any other ball, to its weight. Or, take $\frac{9}{64}$ of the cube of the diameter, for the weight. Or, take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight.

EXAMPLES.

EXAM. 1. The diameter of an iron shot being 6·7 inches, required its weight? Ans. 42·294lb.

EXAM. 2. What is the weight of an iron ball, whose diameter is 5·54 inches? Ans. 24lb nearly.

PROBLEM II.

To find the Weight of a Leaden Ball.

A leaden ball of 1 inch diameter weighs $\frac{3}{14}$ of a lb; therefore as the cube of 1 is to $\frac{3}{14}$, or as 14 is to 3, so is the cube of

of the diameter of a leaden ball, to its weight. Or, take $\frac{3}{14}$ of the cube of the diameter, for the weight, nearly.

EXAMPLES.

EXAM. 1. Required the weight of a leaden ball of 6·6 inches diameter? Ans. 61·606lb.

EXAM. 2. What is the weight of a leaden ball of 5·30 inches diameter? Ans. 32lb nearly.

PROBLEM III.

To find the Diameter of an Iron Ball.

MULTIPLY the weight by $7\frac{1}{9}$, and the cube root of the product will be the diameter.

EXAMPLES.

EXAM. 1. Required the diameter of a 42lb iron ball?
Ans. 6·685 inches.

EXAM. 2. What is the diameter of a 24lb iron ball?
Ans. 5·54 inches.

PROBLEM IV.

To find the Diameter of a Leaden Ball.

MULTIPLY the weight by 14, and divide the product by 3; then the cube root of the quotient will be the diameter.

EXAMPLES.

EXAM. 1. Required the diameter of a 64lb leaden ball?
Ans. 6·684 inches.

EXAM. 2. What is the diameter of an 8lb leaden ball?
Ans. 3·343 inches.

PROBLEM V.

To find the Weight of an Iron Shell.

TAKE $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the shell.

That is, from the cube of the external diameter, take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

EXAMPLES.

EXAMPLES.

EXAM. 1. The outside diameter of an iron shell being 12·8, and the inside diameter 9·1 inches; required its weight?
Ans. 188·941 lb.

EXAM. 2. What is the weight of an iron shell, whose external and internal diameters are 9·8 and 7 inches?
Ans. $84\frac{1}{3}$lb.

PROBLEM VI.

To find how much Powder will fill a Shell.

DIVIDE the cube of the internal diameter, in inches, by 57·3, for the lbs of powder.

EXAMPLES.

EXAM. 1. How much powder will fill the shell whose internal diameter is 9·1 inches? Ans. $13\frac{2}{13}$lb nearly.

EXAM. 2. How much powder will fill a shell whose internal diameter is 7 inches? Ans. 6lb.

PROBLEM VII.

To find how much Powder will fill a Rectangular Box.

FIND the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

EXAMPLES.

EXAM. 1. Required the quantity of powder that will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches? Ans. 60lb.

EXAM. 2. How much powder will fill a cubical box whose side is 12 inches? Ans. $57\frac{3}{5}$lb.

PROBLEM VIII.

To find how much Powder will fill a Cylinder.

MULTIPLY the square of the diameter by the length, then divide by 38·2 for the pounds of powder.

EXAMPLES.

EXAM. 1. How much powder will the cylinder hold, whose diameter is 10 inches, and length 20 inches?
Ans. $52\frac{1}{3}$lb nearly.

EXAM. 2

EXAM. 2. How much powder can be contained in the cylinder whose diameter is 4 inches, and length 12 inches?
Ans. $5\frac{5}{191}$lb.

PROBLEM IX.

To find the Size of a Shell to contain a given Weight of Powder.

MULTIPLY the pounds of powder by 57·3, and the cube root of the product will be the diameter in inches.

EXAMPLES.

EXAM. 1. What is the diameter of a shell that will hold $13\frac{1}{8}$lb of powder? Ans. 9·1 inches.

EXAM. 2. What is the diameter of a shell to contain 6lb of powder? Ans. 7 inches.

PROBLEM X.

To find the Size of a Cubical Box, to contain a given Weight of Powder.

MULTIPLY the weight in pounds by 30, and the cube root of the product will be the side of the box in inches.

EXAMPLES.

EXAM. 1. Required the side of a cubical box, to hold 50lb of gunpowder? Ans. 11·44 inches.

EXAM. 2. Required the side of a cubical box, to hold 400lb of gunpowder? Ans. 22·89 inches.

PROBLEM XI.

To find what Length of a Cylinder will be filled by a given Weight of Gunpowder.

MULTIPLY the weight in pounds by 38·2, and divide the product by the square of the diameter in inches, for the length.

EXAMPLES.

EXAM. 1. What length of a 36-pounder gun, of $6\frac{2}{3}$ inches diameter, will be filled with 12lb of gunpowder?
Ans. 10·314 inches.

EXAM. 2. What length of a cylinder, of 8 inches diameter, may be filled with 20lb of powder? Ans. $11\frac{15}{16}$ inches.

OF

Of the PILING of BALLS and SHELLS.

Iron Balls and Shells are commonly piled by horizontal courses, either in a pyramidical or in a wedge-like form; the base being either an equilateral triangle, or a square, or a rectangle. In the triangle and square, the pile finishes in a single ball; but in the rectangle, it finishes in a single row of balls, like an edge.

In triangular and square piles, the number of horizontal rows, or courses, is always equal to the number of balls in one side of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Also, the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row.

PROBLEM I.

To find the Number of Balls in a Triangular Pile.

Multiply continually together the number of balls in one side of the bottom row, and that number increased by 1, also the same number increased by 2; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{n \,.\, \overline{n+1} \,.\, \overline{n+2}}{6}$ is the number or sum, where n is the number in the bottom row.

EXAMPLES.

Exam. 1. Required the number of balls in a triangular pile, each side of the base containing 30 balls? Ans. 4960.

Exam. 2. How many balls are in the triangular pile, each side of the base containing 20? Ans. 1540.

PROBLEM II.

To find the Number of Balls in a Square Pile.

Multiply continually together the number in one side of the bottom course, that number increased by 1, and double the same number increased by 1; then $\frac{1}{6}$ of the last product will be the answer.

That is, $\frac{n \,.\, \overline{n+1} \,.\, \overline{2n+1}}{6}$ is the number.

EXAMPLES.

EXAM. 1. How many balls are in a square pile of 30 rows? Ans. 9455.

EXAM. 2. How many balls are in a square pile of 20 rows? Ans. 2870.

PROBLEM III.

To find the Number of Balls in a Rectangular Pile.

FROM 3 times the number in the length of the base row, subtract one less than the breadth of the same, multiply the remainder by the same breadth, and the product by one more than the same; and divide by 6 for the answer.

That is, $\frac{b \cdot \overline{b+1} \cdot \overline{3l - b + 1}}{6}$ is the number; where l is the length, and b the breadth of the lowest course.

Note. In all the piles the breadth of the bottom is equal to the number of courses. And in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

EXAMPLES.

EXAM. 1. Required the number of balls in a rectangular pile, the length and breadth of the base row being 46 and 15? Ans. 4960.

EXAM. 2. How many shot are in a rectangular complete pile, the length of the bottom course being 59, and its breadth 20? Ans. 11060.

PROBLEM IV.

To find the Number of Balls in an Incomplete Pile.

FROM the number in the whole pile, considered as complete, subtract the number in the upper pile which is wanting at the top, both computed by the rule for their proper form; and the remainder will be the number in the frustum, or incomplete pile.

EXAMPLES.

EXAM. 1. To find the number of shot in the incomplete triangular pile, one side of the bottom course being 40, and the top course 20? Ans. 10150.

EXAM. 2.

EXAM. 2. How many shot are in the incomplete triangular pile, the side of the base being 24, and of the top 8?

Ans. 2516.

EXAM. 3. How many balls are in the incomplete square pile, the side of the base being 24, and of the top 8?

Ans. 4760.

EXAM. 4. How many shot are in the incomplete rectangular pile, of 12 courses, the length and breadth of the base being 40 and 20? Ans. 6146.

OF DISTANCES BY THE VELOCITY OF SOUND.

BY various experiments it has been found, that sound flies, through the air, uniformly at the rate of about 1142 feet in 1 second of time, or a mile in $4\frac{2}{3}$ or $\frac{14}{3}$ seconds. And therefore, by proportion, any distance may be found corresponding to any given time; namely, multiplying the given time, in seconds, by 1142, for the corresponding distance in feet; or taking $\frac{3}{14}$ of the given time for the distance in miles. Or dividing any given distance by these numbers, to find the corresponding time.

Note. The time for the passage of sound in the interval between seeing the flash of a gun, or lightning, and hearing the report, may be observed by a watch, or a small pendulum. Or, it may be observed by the beats of the pulse in the wrist, counting, on an average, about 70 to a minute for persons in moderate health, or $5\frac{1}{4}$ pulsations to a mile; and more or less according to circumstances.

EXAMPLES.

EXAM. 1. After observing a flash of lightning, it was 12 seconds before the thunder was heard; required the distance of the cloud from whence it came? Ans. $2\frac{4}{7}$ miles.

EXAM. 2. How long, after firing the Tower guns, may the report be heard at Shooter's-Hill, supposing the distance to be 8 miles in a straight line? Ans. $37\frac{1}{3}$ seconds.

EXAM. 3. After observing the firing of a large cannon at a distance, it was 7 seconds before the report was heard; what was its distance? Ans. $1\frac{1}{2}$ mile.

EXAM. 4. Perceiving a man at a distance hewing down a tree with an axe, I remarked that 6 of my pulsations passed between seeing him strike and hearing the report of the

blow; what was the distance between us, allowing 70 pulses to a minute? Ans. 1 mile and 198 yards.

EXAM. 5. How far off was the cloud from which thunder issued, whose report was 5 pulsations after the flash of lightning; counting 75 to a minute? Ans. 1523 yards.

EXAM. 6. If I see the flash of a cannon, fired by a ship in distress at sea, and hear the report 33 seconds after, how far is she off? Ans. $7\frac{1}{14}$ miles.

PRACTICAL EXERCISES IN MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, AND OTHER BRANCHES OF NATURAL PHILOSOPHY.

QUESTION 1. REQUIRED the weight of a cast iron ball of 3 inches diameter, supposing the weight of a cubic inch of the metal to be 0·258lb avoirdupois? Ans. 3·64739lb.

QUEST. 2. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch? Ans. 13·23871b.

QUEST. 3. Being one day ordered to observe how far a battery of cannon was from me, I counted, by my watch, 17 seconds between the time of seeing the flash and hearing the report; what then was the distance? Ans. $3\frac{2}{7}$ miles.

QUEST. 4. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Ans. as 71 to 1 nearly.

QUEST. 5. What difference is there, in point of weight, between a block of marble, containing 1 cubic foot and a half, and another of brass of the same dimensions?

Ans. 496lb 14oz.

QUEST. 6. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three stones laid end to end, now in sight, that measure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, so as to prepare it for moving?

Ans. $683\frac{7}{16}$ tons, the burden of an East-India ship.

QUEST. 7. The battering-ram of Vespasian weighed, suppose 10,000 pounds; and was moved, let us admit, with such

such a velocity, by strength of hand, as to pass through 20 feet in one second of time; and this was found sufficient to demolish the walls of Jerusalem. The question is, with what velocity a 32lb ball must move, to do the same execution? Ans. 6250 feet.

QUEST. 8. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier; but the less moves with 1000 times the velocity of the greater: in what proportion then are the momenta, or forces, with which they moved?

Ans. the less moves with a force 40 times greater.

QUEST. 9. A body, weighing 20lb, is impelled by such a force, as to send it through 100 feet in a second; with what velocity then would a body of 8lb weight move, if it were impelled by the same force? Ans. 250 feet per second.

QUEST. 10. There are two bodies, the one of which weighs 100lb, the other 60; but the less body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which these bodies move, is required?

Ans. the velocity of the greater to that of the less, as 3 to 40.

QUEST. 11. There are two bodies, the greater contains 8 times the quantity of matter in the less, and is moved with a force 48 times greater: the ratio of the velocities of these two bodies is required?

Ans. the greater is to the less, as 6 to 1.

QUEST. 12. There are two bodies, one of which moves 40 times swifter than the other; but the swifter body has moved only one minute, whereas the other has been in motion 2 hours: the ratio of the spaces described by these two bodies is required?

Ans. the swifter is to the slower, as 1 to 3.

QUEST. 13. Supposing one body to move 30 times swifter than another, as also the swifter to move 12 minutes, the other only 1: what difference will there be between the spaces described by them, supposing the last has moved 5 feet? Ans. 1795 feet.

QUEST. 14. There are two bodies, the one of which has passed over 50 miles, the other only 5; and the first had moved with 5 times the celerity of the second; what is the ratio of the times they have been in describing those spaces?

Ans. as 2 to 1.

QUEST. 15. If a lever, 40 effective inches long, will, by a certain power thrown successively on it, in 13 hours, raise a weight 104 feet; in what time will two other levers, each

each 18 effective inches long, raise an equal weight 75 feet? Ans. 10 hours $8\frac{1}{3}$ minutes.

QUEST. 16. What weight will a man be able to raise, who presses with the force of a hundred and a half, on the end of an equipoised handspike, 100 inches long, meeting with a convenient prop exactly $7\frac{1}{2}$ inches from the lower end of the machine? Ans. 2072lb.

QUEST. 17. A weight of $1\frac{1}{2}$lb, laid on the shoulder of a man, is no greater burden to him than its absolute weight, or 24 ounces: what difference will he feel, between the said weight applied near his elbow, at 12 inches from the shoulder, and in the palm of his hand, 28 inches from the same; and how much more must his muscles then draw, to support it at right angles, that is, having his arm stretched right out? Ans. 24lb avoirdupois.

QUEST. 18. What weight hung on at 70 inches from the centre of motion of a steel-yard, will balance a small gun of $9\frac{1}{2}$ cwt, freely suspended at 2 inches distance from the said centre on the contrary side? Ans. $30\frac{2}{5}$lb.

QUEST. 19. It is proposed to divide the beam of a steel-yard, or to find the points of division where the weights of 1, 2, 3, 4, &c, lb, on the one side, will just balance a constant weight of 95lb at the distance of 2 inches on the other side of the fulcrum; the weight of the beam being 10lb, and its whole length 36 inches?

Ans. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, $2\frac{8}{11}$, $2\frac{1}{2}$, &c.

QUEST. 20. Two men carrying a burden of 200lb weight between them, hung on a pole, the ends of which rest on their shoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet? Ans. 125lb and 75lb.

QUEST. 21. If, in a pair of scales, a body weigh 90lb in one scale, and only 40lb in the other; required its true weight, and the proportion of the lengths of the two arms of the balance beam, on each side of the point of suspension?

Ans. the weight 60lb, and the proportion 3 to 2.

QUEST. 22. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For instance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itself when the prop is 13 feet from the less end; but removing the prop a foot nearer to the said end, it takes a man's weight of 210lb, standing on the less end,

end, to hold it in equilibrium. Required the weight of the tree? Ans. 2520lb.

QUEST. 23. If AB be a cane or walking-stick, 40 inches long, suspended by a string SD fastened to the middle point D: now a body being hung on at E, 6 inches distance from D, is balanced by a weight of 2lb, hung on at the larger end A; but removing the body to F, one inch nearer to D, the 2lb weight on the other side is moved to G, within 8 inches of D, before the cane will rest in equilibrio. Required the weight of the body? Ans. 24lb.

QUEST. 24. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B: what will be the ratio of two weights P, Q, in equilibrio on the planes, in all positions of them: and what will be the altitude BD of the angle B above the horizontal plane AC, when this is 50 inches long?

Ans. BD = 24; and P to Q as AB to BC, or as 3 to 4.

QUEST. 25. A lever, of 6 feet long, is fixed at right angles in a screw, whose threads are one inch asunder, so that the lever turns just once round in raising or depressing the screw one inch. If then this lever be urged by a weight or force of 50lb, with what force will the screw press?

Ans. 22619½lb.

QUEST. 26. If a man can draw a weight of 150lb up the side of a perpendicular wall, of 20 feet high; what weight will he be able to raise along a smooth plank of 30 feet long, laid aslope from the top of the wall? Ans. 225lb.

QUEST. 27. If a force of 150lb be applied on the head of a rectangular wedge, its thickness being 2 inches, and the length of its side 12 inches; what weight will it raise or balance perpendicular to its side? Ans. 900lb.

QUEST. 28. If a round pillar of 30 feet diameter be raised on a plane, inclined to the horizon in an angle of 75°, or the shaft inclining 15 degrees out of the perpendicular; what length will it bear before it overset?

Ans. $30\,(2+\sqrt{3})$ or 111·9615 feet.

QUEST. 29. If the greatest angle at which a bank of natural earth will stand, be 45°; it is proposed to determine what thickness an upright wall of stone must be made throughout, just to support a bank of 12 feet high; the specific gravity of the stone being to that of earth, as 5 to 4.

Ans. $\frac{48}{5}\sqrt{\frac{1}{5}}$, or 4·29325 feet.

QUEST. 30. If the stone wall be made like a wedge, or having its upright section a triangle, tapering to a point at

top,

top, but its side next the bank of earth perpendicular to the horizon; what is its thickness at the bottom, so as to support the same bank? Ans. $12\sqrt{\frac{1}{5}}$, or 5·36656 feet.

QUEST. 31. But if the earth will only stand at an angle of 30 degrees to the horizontal line; it is required to determine the thickness of wall in both the preceding cases?

Ans. the breadth of the rectangle $12\sqrt{\frac{1}{5}}$, or 5·36656, but the base of the triangular bank $12\sqrt{\frac{3}{10}}$, or 6·53667.

QUEST. 32. To find the thickness of an upright rectangular wall, necessary to support a body of water; the water being 10 feet deep, and the wall 12 feet high; also the specific gravity of the wall to that of the water, as 11 to 7.

Ans. 4·204374 feet.

QUEST. 33. To determine the thickness of the wall at the bottom, when the section of it is triangular, and the altitudes as before. Ans. 5·1492866 feet.

QUEST. 34. Supposing the distance of the earth from the sun to be 95 millions of miles; I would know at what distance from him another body must be placed, so as to receive light and heat quadruple to that of the earth.

Ans. at half the distance, or $47\frac{1}{2}$ millions.

QUEST. 35. If the mean distance of the sun from us be 106 of his diameters; how much hotter is it at the surface of the sun, than under our equator?

Ans. 11236 times hotter.

QUEST. 36. The distance between the earth and the sun being accounted 95 millions of miles, and between Jupiter and the sun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth, is required?

Ans. $\frac{361}{9801}$, or nearly $\frac{1}{27}$ of the earth's light and heat.

QUEST. 37. A certain body on the surface of the earth weighs a cwt, or 112lb; the question is whither this body must be carried, that it may weigh only 10lb?

Ans. either at 3·3466 semi-diameters, or $\frac{5}{56}$ of a semi-diameter, from the centre.

QUEST. 38. If a body weigh 1 pound, or 16 ounces, on the surface of the earth; what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles?

Ans. 15oz. $9\frac{1}{8}$dr. nearly.

QUEST. 39. Whereabouts, in the line between the earth and moon, is their common centre of gravity; supposing the earth's diameter to be 7930 miles, and the moon's 2160; also the

the density of the former to that of the latter, as 99 to 68, or as 10 to 7 nearly, and their mean distance 30 of the earth's diameters?

Ans. at $\frac{105}{251}$ parts of a diameter from the earth's centre, or $\frac{41}{502}$ parts of a diameter, or 648 miles below the surface.

QUEST. 40. Whereabouts, between the earth and moon, are their attractions equal to each other? Or where must another body be placed, so as to remain suspended in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way? Their dimensions being as in the last question.

$$\left.\begin{array}{l}\text{Ans. From the earth's centre } 26\tfrac{9}{11}\\ \text{From the moon's centre } 3\tfrac{2}{11}\end{array}\right\}\text{ of the earth's diameters.}$$

QUEST. 41. Suppose a stone dropt into an abyss, should be stopped at the end of the 11th second after its delivery; what space would it have gone through?

Ans. $1946\frac{1}{12}$ feet.

QUEST. 42. What is the difference between the depths of two wells, into each of which should a stone be dropped at the same instant, the one will strike the bottom at 6 seconds, the other at 10? Ans. $1029\frac{1}{3}$ feet.

QUEST. 43. If a stone be $19\frac{1}{2}$ seconds in descending from the top of a precipice to the bottom, what is its height?

Ans. $6115\frac{11}{16}$ feet.

QUEST. 44. In what time will a musket ball, dropped from the top of Salisbury steeple, said to be 400 feet high, reach the bottom? Ans. 5 seconds nearly.

QUEST. 45. If a heavy body be observed to fall through 100 feet in the last second of time, from what height did it fall, and how long was it in motion?

Ans. time $3\frac{235}{386}$ sec. and height $209\frac{4273}{9264}$ feet.

QUEST. 46. A stone being let fall into a well, it was observed that, after being dropped, it was 10 seconds before the sound of the fall at the bottom reached the ear. What is the depth of the well? Ans. 1270 feet nearly.

QUEST. 47. It is proposed to determine the length of a pendulum vibrating seconds, in the latitude of London, where a heavy body falls through $16\frac{1}{12}$ feet in the first second of time? Ans. 39·11 inches.

By experiment this length is found to be $39\frac{1}{8}$ inches.

QUEST. 48.

QUEST. 48. What is the length of a pendulum vibrating in 2 seconds; also in half a second, and in a quarter second?

Ans. the 2 second pendulum $156\frac{1}{2}$
the $\frac{1}{2}$ second pendulum $9\frac{25}{32}$
the $\frac{1}{4}$ second pendulum $2\frac{57}{128}$ inches.

QUEST. 49. What difference will there be in the number of vibrations, made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time? Ans. $2692\frac{1}{2}$.

QUEST 50. Observed that while a stone was descending, to measure the depth of a well, a string and plummet, that from the point of suspension, or the place where it was held, to the centre of oscillation, measured just 18 inches, had made 8 vibrations, when the sound from the bottom returned. What was the depth of the well?

Ans. 412·61 feet.

QUEST. 51. If a ball vibrate in the arch of a circle, 10 degrees on each side of the perpendicular; or a ball roll down the lowest 10 degrees of the arch; required the velocity at the lowest point? the radius of the circle, or length of the pendulum, being 20 feet. Ans. 4·4213 feet per second.

QUEST. 52. If a ball descend down a smooth inclined plane, whose length is 100 feet, and altitude 10 feet; how long will it be in descending, and what will be the last velocity?

Ans. the veloc. 25·364 feet per sec. and time 7·8852 sec.

QUEST. 53. If a cannon ball, of 1lb weight, be fired against a pendulous block of wood, and, striking the centre of oscillation, cause it to vibrate an arc whose chord is 30 inches; the radius of that arc, or distance from the axis to the lowest point of the pendulum, being 118 inches, and the pendulum vibrating in small arcs 40 oscillations per minute. Required the velocity of the ball, and the velocity of the centre of oscillation of the pendulum, at the lowest point of the arc; the whole weight of the pendulum being 500lb?

Ans. veloc. ball 1956·6054 feet per sec.
and veloc. cent. oscil. 3·9054 feet per sec.

QUEST. 54. How deep will a cube of oak sink in common water; each side of the cube being 1 foot?

Ans. $11\frac{1}{10}$ inches.

QUEST. 55. How deep will a globe of oak sink in water; the diameter being 1 foot? Ans. 9·9867 inches.

QUEST.

QUEST. 56. If a cube of wood, floating in common water, have three inches of it dry above the water, and $4\frac{8}{103}$ inches dry when in sea-water; it is proposed to determine the magnitude of the cube, and what sort of wood it is made of? Ans. the wood is oak, and each side 40 inches.

QUEST. 57. An irregular piece of lead ore weighs, in air 12 ounces, but in water only 7; and another fragment weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative densities, or specific gravities?

Ans. as 145 to 132.

QUEST. 58. An irregular fragment of glass, in the scale, weighs 171 grains, and another of magnet 102 grains; but in water the first fetches up no more than 120 grains, and the other 79: what then will their specific gravities turn out to be? Ans. glass to magnet as 3933 to 5202 or nearly as 10 to 13.

QUEST. 59. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that substituting part silver was only a proper perquisite; which taking air, Archimedes was appointed to examine it; who, on putting it into a vessel of water, found it raised the fluid 8·2245 cubic inches: and having discovered that the inch of gold more critically weighed 10·86 ounces, and that of silver but 5·35 ounces, he found by calculation what part of the king's gold had been changed. And you are desired to repeat the process.

Ans. 28·8 ounces.

QUEST. 60. Supposing the cubic inch of common glass weigh 1·4921 ounces troy, the same of sea-water ·59542, and of brandy ·5368; then a seaman having a gallon of this liquor in a glass bottle, which weighs 3·84lb out of water, and, to conceal it from the officers of the customs, throws it overboard. It is proposed to determine, if it will sink, how much force will just buoy it up?

Ans. 14·1496 ounces.

QUEST. 61. Another person has half an anker of brandy, of the same specific gravity as in the last question; the wood of the cask suppose measures $\frac{1}{8}$ of a cubic foot; it is proposed to assign what quantity of lead is just requisite to keep the cask and liquor under water? Ans. 89·743 ounces.

QUEST. 62. Suppose, by measurement, it be found that a man-of-war, with its ordnance, rigging, and appointments, sinks

sinks so deep as to displace 50000 cubic feet of fresh water; what is the whole weight of the vessel?

Ans. $1395\frac{1}{10}$ tons.

QUEST. 63. It is required to determine what would be the height of the atmosphere, if it were every where of the same density as at the surface of the earth, when the quicksilver in the barometer stands at 30 inches; and also, what would be the height of a water barometer at the same time?

Ans. height of the air $29166\frac{2}{3}$ feet, or 5·5240 miles, height of water 35 feet.

QUEST. 64. With what velocity would each of those three fluids, viz. quicksilver, water, and air, issue through a small orifice in the bottom of vessels, of the respective heights of 30 inches, 35 feet, and 5·5240 miles, estimating the pressure by the whole altitudes, and the air rushing into a vacuum?

Ans. the veloc. of quicksilver 12·681 feet.
the veloc. of water - 47·447
the veloc. of air - - 1369.8

QUEST. 65. A very large vessel of 10 feet high (no matter what shape) being kept constantly full of water, by a large supplying cock at the top; if 9 small circular holes, each $\frac{1}{2}$ of an inch diameter, be opened in its perpendicular side at every foot of the depth: it is required to determine the several distances to which they will spout on the horizontal plane of the base, and the quantity of water discharged by all of them in 10 minutes?

Ans. the distances are
$\sqrt{36}$ or 6·00000
$\sqrt{64}$ - 8·00000
$\sqrt{84}$ - 9·16515
$\sqrt{96}$ - 9·79796
$\sqrt{100}$ - 10·00000
$\sqrt{96}$ - 9·79796
$\sqrt{84}$ - 9·16515
$\sqrt{64}$ - 8·00000
$\sqrt{36}$ - 6·00000

and the quantity discharged in 10 min, 123.8849 gallons.

Note. In this solution, the velocity of the water is supposed to be equal to that which is acquired by a heavy body in falling through the whole height of the water above the orifice, and that it is the same in every part of the holes.

QUEST.

QUEST. 66. If the inner axis of a hollow globe of copper, exhausted of air, be 100 feet; what thickness must it be of, that it may just float in the air?

Ans. ·02688 of an inch thick.

QUEST. 67. If a spherical balloon of copper, of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of $\frac{1}{10}$ of the gravity of common air, what weight will just balance it, and prevent it from rising up into the atmosphere? Ans. 21273lb.

QUEST. 68. If a glass tube, 36 inches long, close at top, be sunk perpendicularly into water, till its lower or open end be 30 inches below the surface of the water; how high will the water rise within the tube, the quicksilver in the common barometer at the same time standing at $29\frac{1}{2}$ inches?

Ans. 2·26545 inches.

QUEST. 69. If a diving bell, of the form of a parabolic conoid, be let down into the sea to the several depths of 5, 10, 15, and 20 fathoms; it is required to assign the respective heights to which the water will rise within it: its axis and the diameter of its base being each 8 feet, and the quicksilver in the barometer standing at 30·9 inches?

Ans. at	5 fathoms deep the water rises	2·03546 feet.
at 10	- - -	3·06393
at 15	- - -	3·70267
at 20	- - -	4·14653

THE

sinks so deep as to displace 50000 cubic
water; what is the whole weight of the ve

QUEST. 63. It is required to deter-
height of the atmosphere, if it
same density as at the surface of
silver in the barometer stand
would be the height of a w

Ans. height of th
height of

QUEST. 64. With
three fluids, viz.
a small orifice in
heights of 30 i
the pressure
into a vacu

FLUXIONS.

AND PRINCIPLES.

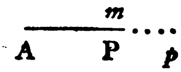
...rine of Fluxions, magnitudes or quan-
...considered, not as made up of a number
...as generated by continued motion, by
...they increase or decrease. As, a line by
...point; a surface by the motion of a line;
...the motion of a surface. So likewise, time
...ered as represented by a line, increasing uni-
...the motion of a point. And quantities of all
...ever, which are capable of increase and decrease,
...like manner be represented by geometrical magni-
...conceived to be generated by motion.

2. Any quantity thus generated, and variable, is called a Fluent, or a Flowing Quantity. And the rate or proportion according to which any flowing quantity increases, at any position or instant, is the Fluxion of the said quantity, at that position or instant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increased in a given time, with the generating celerity uniformly continued during that time.

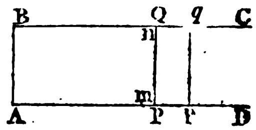
3. The small quantities that are actually generated, produced, or described, in any small given time, and by any continued motion, either uniform or variable, are called Increments.

4. Hence, if the motion of increase be uniform, by which increments are generated, the increments will in that case be proportional, or equal, to the measures of the fluxions: but if the motion of increase be accelerated, the increment so generated, in a given finite time, will exceed the fluxion: and if it be a decreasing motion, the increment, so generated, will be less than the fluxion. But if the time be indefinitely small, so that the motion be considered as uniform for that instant; then these nascent increments will always be proportional, or equal, to the fluxions, and may be substituted instead of them, in any calculation.

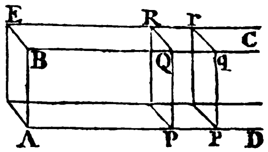
5. To



5. To illustrate these definitions: Suppose a point *m* be conceived to move from the position A, and to generate a line AP, by a motion any how regulated; and suppose the celerity of the point *m*, at any position P, to be such, as would, if from thence it should become or continue uniform, be sufficient to cause the point to describe, or pass uniformly over, the distance P*p*, in the given time allowed for the fluxion: then will the said line P*p* represent the fluxion of the fluent, or flowing line, AP, at that position.



6. Again, suppose the right line *mn* to move, from the position AB, continually parallel to itself, with any continued motion, so as to generate the fluent or flowing rectangle ABQP, while the point *m* describes the line AP: also, let the distance P*p* be taken, as before, to express the fluxion of the line or base AP; and complete the rectangle PQ*qp*. Then, like as P*p* is the fluxion of the line AP, so is P*q* the fluxion of the flowing parallelogram AQ; both these fluxions, or increments, being uniformly described in the same time.

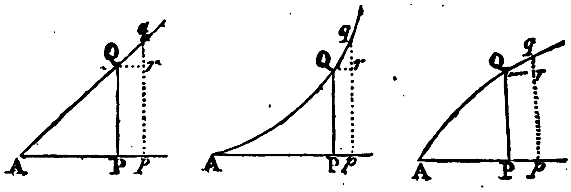


7. In like manner, if the solid AERP be conceived to be generated by the plane PQR, moving from the position ABE, always parallel to itself, along the line AD; and if P*p* denote the fluxion of the line AP: Then, like as the rectangle PQ*qp*, or PQ × P*p*, denotes the fluxion of the flowing rectangle ABQP, so also shall the fluxion of the variable solid, or prism ABERQP, be denoted by the prism PQR*rqp*, or the plane PR × P*p*. And, in both these last two cases, it appears that the fluxion of the generated rectangle, or prism, is equal to the product of the generating line, or plane, drawn into the fluxion of the line along which it moves.

8. Hitherto the generating line, or plane, has been considered as of a constant and invariable magnitude; in which case the fluent, or quantity generated, is a rectangle, or a prism, the former being described by the motion of a line, and the latter by the motion of a plane. So, in like manner are other figures, whether plane or solid, conceived to be de-

scribed

scribed by the motion of a Variable Magnitude, whether it be a line or a plane. Thus, let a variable line PQ be carried by a parallel motion along AP; or while a point P is carried along, and describes the line AP, suppose another point



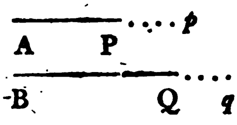
Q to be carried by a motion perpendicular to the former, and to describe the line PQ: let *pq* be another position of PQ, indefinitely near to the former; and draw Q*r* parallel to AP. Now in this case there are several fluents, or flowing quantities, with their respective fluxions: namely, the line or fluent AP, the fluxion of which is P*p* or Q*r*; the line or fluent PQ, the fluxion of which is *rq*; the curve or oblique line AQ, described by the oblique motion of the point Q, the fluxion of which is Q*q*; and lastly, the surface APQ, described by the variable line PQ, the fluxion of which is the rectangle PQ*rp*, or PQ × P*p*. In the same manner may any solid be conceived to be described, by the motion of a variable plane parallel to itself, substituting the variable plane for the variable line; in which case the fluxion of the solid, at any position, is represented by the variable plane, at that position, drawn into the fluxion of the line along which it is carried.

9. Hence then it follows in general, that the fluxion of any figure, whether plane or solid, at any position, is equal to the section of it, at that position, drawn into the fluxion of the axis, or line along which the variable section is supposed to be perpendicularly carried; that is, the fluxion of the figure AQP, is equal to the plane PQ × P*p*, when that figure is a solid, or to the ordinate PQ × P*p*, when the figure is a surface.

10. It also follows from the same premises, that in any curve, or oblique line AQ, whose absciss is AP, and ordinate is PQ, the fluxions of these three form a small right-angled plane triangle Q*qr*; for Q*r* = P*p* is the fluxion of the absciss AP, *qr* the fluxion of the ordinate PQ, and Q*q* the fluxion of the curve or right line AQ. And consequently that, in any curve, the square of the fluxion of the curve, is equal to the sum

sum of the squares of the fluxions of the absciss and ordinate, when these two are at right angles to each other.

11. From the premises it also appears, that contemporaneous fluents, or quantities that flow or increase together, which are always in a constant ratio to each other, have their fluxions also in the same constant ratio, at every position. For, let AP and BQ be two contemporaneous fluents, described in the same time by the motion of the points P and Q, the contemporaneous positions being P, Q, and p, q; and let AP be to BQ, or Ap to Bq, constantly in the ratio of 1 to n.



Then - - - - - - is $n \times \text{AP} = \text{BQ}$,
and $n \times \text{A}p = \text{B}q$;
therefore, by subtraction, $n \times \text{P}p = \text{Q}q$;
that is, the fluxion - Pp : fluxion Qq :: 1 : n,
the same as the fluent AP : fluent BQ :: 1 : n;
or, the fluxions and fluents are in the same constant ratio.

But if the ratio of the fluents be variable, so will that of the fluxions be also, though not in the same variable ratio with the former, at every position.

NOTATION, &c.

12. To apply the foregoing principles to the determination of the fluxions of algebraic quantities, by means of which those of all other kinds are assigned, it will be necessary first to premise the notation commonly used in this science, with some observations. As, first, that the final letters of the alphabet z, y, x, u, &c, are used to denote variable or flowing quantities; and the initial letters a, b, c, d, &c, to denote constant or invariable ones: Thus, the variable base AP of the flowing rectangular figure ABQP, in art. 6, may be represented by x; and the invariable altitude PQ, by a: also, the variable base or absciss AP, of the figures in art. 8, may be represented by x, the variable ordinate PQ, by y; and the variable curve or line AQ, by z.

Secondly, that the fluxion of a quantity denoted by a single letter, is represented by the same letter with a point over it: Thus, the fluxion of x is expressed by $\dot{x}$, the fluxion of y by $\dot{y}$, and the fluxion of z by $\dot{z}$. As to the fluxions of constant or invariable quantities, as of a, b, c, &c, they are equal to nothing, because they do not flow or change their magnitude.

Thirdly, that the increments of variable or flowing quantities, are also denoted by the same letters with a small ′ over them: Thus, the increments of x, y, z, are x', y', z'.

13. From these notations, and the foregoing principles, the quantities, and their fluxions, there considered, will be denoted as below. Thus, in all the foregoing figures, put

the variable or flowing line - - AP $= x$,
in art. 6, the constant line - - PQ $= a$,
in art. 8, the variable ordinate - PQ $= y$,
also, the variable line or curve - AQ $= z$:

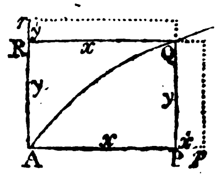
Then shall the several fluxions be thus represented, namely,

$\dot{x} =$ Pp the fluxion of the line AP,
$a\dot{x} =$ PQqp the fluxion of ABQP in art. 6,
$y\dot{x} =$ PQrp the fluxion of APQ in art. 8,
$\dot{z} =$ Qq $= \sqrt{(\dot{x}^2 + \dot{y}^2)}$ the fluxion of AQ; and
$a\dot{x} =$ Pr the fluxion of the solid in art. 7, if a denote the constant generating plane PQR; also,
$nx =$ BQ in the figure to art. 11, and
$n\dot{x} =$ Qq the fluxion of the same.

14. The principles and notation being now laid down, we may proceed to the practice and rules of this doctrine; which consists of two principal parts, called the Direct and Inverse Method of Fluxions; namely, the direct method, which consists in finding the fluxion of any proposed fluent or flowing quantity; and the inverse method, which consists in finding the fluent of any proposed fluxion. As to the former of these two problems, it can always be determined, and that in finite algebraic terms; but the latter, or finding of fluents, can only be effected in some certain cases, except by means of infinite series.—First then, of

THE DIRECT METHOD OF FLUXIONS.

To find the Fluxion of the Product or Rectangle of two Variable Quantities.



15. Let ARQP, $= xy$, be the flowing or variable rectangle, generated by two lines PQ and RQ, moving always perpendicular to each other, from the positions AR and AP; denoting the one by x, and the other by y; supposing x and y to be so related, that the curve line AQ may always pass through the intersection Q of those lines, or the opposite angle of the rectangle.

Now,

Now, the rectangle consists of the two trilinear spaces APQ, ARQ, of which, the

fluxion of the former is PQ × Pp, or $y\dot{x}$,

that of the latter is - RQ × Rr, or $x\dot{y}$, by art. 8;

therefore the sum of the two $\dot{x}y + x\dot{y}$, is the fluxion of the whole rectangle xy or ARQP.

The Same Otherwise.

16. Let the sides of the rectangle x and y, by flowing, become $x + x'$ and $y + y'$: then the product of these two, or $xy + xy' + yx' + x'y'$ will be the new or contemporaneous value of the flowing rectangle PR or xy: subtract the one value from the other, and the remainder, $xy' + yx' + x'y'$, will be the increment generated in the same time as x' or y'; of which the last term $x'y'$ is nothing, or indefinitely small, in respect of the other two terms, because x' and y' are indefinitely small in respect of x and y; which term being therefore omitted, there remains $xy' + yx'$ for the value of the increment; and hence, by substituting $\dot{x}$ and $\dot{y}$ for x' and y', to which they are proportional, there arises $x\dot{y} + y\dot{x}$ for the true value of the fluxion of xy; the same as before.

17. Hence may be easily derived the fluxion of the powers and products of any number of flowing or variable quantities whatever; as of xyz, or $uxyz$, or $vuxyz$, &c. And first, for the fluxion of xyz: put $p = xy$, and the whole given fluent $xyz = q$, or $q = xyz = pz$. Then, taking the fluxions of $q = pz$, by the last article, they are $\dot{q} = \dot{p}z + p\dot{z}$; but $p = xy$, and so $\dot{p} = \dot{x}y + x\dot{y}$ by the same article; substituting therefore these values of p and $\dot{p}$ instead of them, in the value of $\dot{q}$, this becomes $\dot{q} = \dot{x}yz + x\dot{y}z + xy\dot{z}$, the fluxion of xyz required; which is therefore equal to the sum of the products, arising from the fluxion of each letter, or quantity, multiplied by the product of the other two.

Again, to determine the fluxion of $uxyz$, the continual product of four variable quantities; put this product, namely $uxyz$, or $qu = r$, where $q = xyz$ as above. Then, taking the fluxions by the last article, $\dot{r} = \dot{q}u + q\dot{u}$; which, by substituting for q and $\dot{q}$ their values as above, becomes - - - $\dot{r} = \dot{u}xyz + u\dot{x}yz + ux\dot{y}z + uxy\dot{z}$, the fluxion of $uxyz$ as required: consisting of the fluxion of each quantity, drawn into the products of the other three.

 In

In the very same manner it is found, that the fluxion of $vuxyz$ is $\dot{v}uxyz + v\dot{u}xyz + vu\dot{x}yz + vux\dot{y}z + vuxy\dot{z}$; and so on, for any number of quantities whatever; in which it is always found, that there are as many terms as there are variable quantities in the proposed fluent; and that these terms consist of the fluxion of each variable quantity, multiplied by the product of all the rest of the quantities.

18. Hence is easily derived the fluxion of any power of a variable quantity, as of x^2, or x^3, or x^4, &c. For, in the product or rectangle xy, if $x = y$, then is $xy = xx$ or x^2, and also its fluxion $\dot{x}y + x\dot{y} = \dot{x}x + x\dot{x}$ or $2x\dot{x}$, the fluxion of x^2.

Again, if all the three x, y, z be equal; then is the product of the three $xyz = x^3$; and consequently its fluxion $\dot{x}yz + x\dot{y}z + xy\dot{z} = \dot{x}xx + x\dot{x}x + xx\dot{x}$ or $3x^2\dot{x}$, the fluxion of x^3.

In the same manner, it will appear that
the fluxion of x^4 is $= 4x^3\dot{x}$, and
the fluxion of x^5 is $= 5x^4\dot{x}$ and, in general,
the fluxion of x^n is $= nx^{n-1}\dot{x}$;
where n is any positive whole number whatever.
That is, the fluxion of any positive integral power, is equal to the fluxion of the root $(\dot{x})$, multiplied by the exponent of the power (n), and by the power of the same root whose index is less by 1, (x^{n-1}).

And thus, the fluxion of $a + cx$ being $c\dot{x}$,
that of $(a + cx)^2$ is $2c\dot{x} \times (a + cx)$ or $2ac\dot{x} + 2c^2x\dot{x}$,
that of $(a + cx^2)^2$ is $4cx\dot{x} \times (a + cx^2)$ or $4acx\dot{x} + 4c^2x^3\dot{x}$,
that of $(x^2 + y^2)^2$ is $(4x\dot{x} + 4y\dot{y}) \times (x^2 + y^2)$,
that of $(x + cy^2)^3$ is $(3\dot{x} + 6cy\dot{y}) \times (x + cy^2)^2$.

19. From the conclusions in the same article, we may also derive the fluxion of any fraction, or the quotient of one variable quantity divided by another, as of $\frac{x}{y}$. For, put the quotient or fraction $\frac{x}{y} = q$; then, multiplying by the denominator, $x = qy$; and, taking the fluxions,

$\dot{x} = \dot{q}y + q\dot{y}$, or $\dot{q}y = \dot{x} - q\dot{y}$; and, by division,

$\dot{q} = \frac{\dot{x}}{y} - \frac{q\dot{y}}{y} =$ (by substituting the value of q, or $\frac{x}{y}$),

$\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{\dot{x}y - x\dot{y}}{y^2}$, the fluxion of $\frac{x}{y}$, as required.

That is, the fluxion of any fraction, is equal to the fluxion of the numerator drawn into the denominator, minus the fluxion of the denominator drawn into the numerator, and the remainder divided by the square of the denominator. So that the fluxion of $\frac{ax}{y}$ is $a \times \frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{a\dot{x}y - ax\dot{y}}{y^2}$.

20. Hence too is easily derived the fluxion of any negative integer power of a variable quantity, as of x^{-n}, or $\frac{1}{x^n}$, which is the same thing. For here the numerator of the fraction is 1, whose fluxion is nothing; and therefore, by the last article, the fluxion of such a fraction, or negative power, is barely equal to minus the fluxion of the denominator, divided by the square of the said denominator. That is, the fluxion of x^{-n}, or $\frac{1}{x^n}$ is $-\frac{nx^{n-1}\dot{x}}{x^{2n}}$ or $-\frac{n\dot{x}}{x^{n+1}}$ or $-nx^{-n-1}\dot{x}$; or the fluxion of any negative integer power of a variable quantity, as x^{-n}, is equal to the fluxion of the root, multiplied by the exponent of the power, and by the next power less by 1; the same rule as for positive powers.

The same thing is otherwise obtained thus: Put the proposed fraction, or quotient $\frac{1}{x^n} = q$; then is $qx^n = 1$; and, taking the fluxions, we have

$\dot{q}x^n + qnx^{n-1}\dot{x} = 0$; hence $\dot{q}x^n = -qnx^{n-1}\dot{x}$; divide by x^n, then $\dot{q} = -\frac{qn\dot{x}}{x} =$ (by substituting $\frac{1}{x^n}$ for q), $\frac{-n\dot{x}}{x^{n+1}}$ or $= -nx^{-n-1}\dot{x}$; the same as before.

Hence the fluxion of x^{-1} or $\frac{1}{x}$ is $-x^{-2}\dot{x}$, or $-\frac{\dot{x}}{x^2}$,

that of x^{-2} or $\frac{1}{x^2}$ is $-2x^{-3}\dot{x}$ or $-\frac{2\dot{x}}{x^3}$,

that of x^{-3} or $\frac{1}{x^3}$ is $-3x^{-4}\dot{x}$ or $-\frac{3\dot{x}}{x^4}$,

that of ax^{-4} or $\frac{a}{x^4}$ is $-4ax^{-5}\dot{x}$ or $-\frac{4a\dot{x}}{x^5}$,

that of $(a+x)^{-1}$ or $\frac{1}{a+x}$ is $-(a+x)^{-2}\dot{x}$ or $-\frac{\dot{x}}{(a+x)^2}$,

that of $c(a+3x^2)^{-2}$ or $\frac{c}{(a+3x^2)^2}$ is $-12cx\dot{x} \times (a+3x^2)^{-3}$,

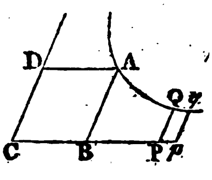
or $-\frac{12cx\dot{x}}{(a+3x^2)^3}$.

21. Mus

21. Much in the same manner is obtained the fluxion of any fractional power of a fluent quantity, as of $x^{\frac{m}{n}}$, or $\sqrt[n]{x^m}$.

For, put the proposed quantity $x^{\frac{m}{n}} = q$; then, raising each side to the n power, gives $x^m = q^n$;
taking the fluxions, gives $mx^{m-1}\dot{x} = nq^{n-1}\dot{q}$; then
dividing by nq^{n-1}, gives $\dot{q} = \frac{mx^{m-1}\dot{x}}{nq^{n-1}} = \frac{mx^{m-1}\dot{x}}{nx^{m-\frac{m}{n}}} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$.
Which is still the same rule, as before, for finding the fluxion of any power of a fluent quantity, and which therefore is general, whether the exponent be positive or negative, integral or fractional. And hence the fluxion of $ax^{\frac{3}{2}}$ is $\frac{3}{2}ax^{\frac{1}{2}}\dot{x}$;
that of $ax^{\frac{1}{2}}$ is $\frac{1}{2}ax^{\frac{1}{2}-1}\dot{x} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x} = \frac{a\dot{x}}{2x^{\frac{1}{2}}} = \frac{a\dot{x}}{2\sqrt{x}}$; and that of
$\sqrt{(a^2-x^2)}$ or $(a^2-x^2)^{\frac{1}{2}}$ is $\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}} \times -2x\dot{x} = \frac{-x\dot{x}}{(\sqrt{a^2-x^2})}$.

22. Having now found out the fluxions of all the ordinary forms of algebraical quantities; it remains to determine those of logarithmic expressions; and also of exponential ones, that is, such powers as have their exponents variable or flowing quantities. And first, for the fluxion of Napier's, or the hyperbolic logarithm.



23. Now, to determine this from the nature of the hyperbolic spaces. Let A be the principal vertex of an hyperbola, having its asymptotes CD, CP, with the ordinates DA, BA, PQ, &c, parallel to them. Then, from the nature of the hyperbola and of logarithms, it is known, that any space ABPQ is the log. of the ratio of CB to CP, to the modulus ABCD. Now, put $1 =$ CB or BA the side of the square or rhombus DB; $m =$ the modulus, or CB $\times$ BA; or area of DB, or sine of the angle C to the radius 1; also the absciss CP $= x$, and the ordinate PQ $= y$. Then, by the nature of the hyperbola, CP $\times$ PQ is always equal to DB, that is, $xy = m$; hence $y = \frac{m}{x}$, and the fluxion of the space, $\dot{x}y$ is $\frac{m\dot{x}}{x} =$ PQqp the fluxion of the log. of x, to the modulus m. And, in the hyperbolic logarithms, the modulus m being 1, therefore,

fore $\frac{\dot{x}}{x}$ is the fluxion of the hyp. log. of x; which is therefore equal to the fluxion of the quantity, divided by the quantity itself.

Hence the fluxion of the hyp. log.

$$\text{of } 1 + x \text{ is } \frac{\dot{x}}{1 + x},$$
$$\text{of } 1 - x \text{ is } \frac{-\dot{x}}{1 - x},$$
$$\text{of } x + z \text{ is } \frac{\dot{x} + \dot{z}}{x + z},$$
$$\text{of } \frac{a + x}{a - x} \text{ is } \frac{\dot{x}(a - x) + \dot{x}(a + x)}{(a - x)^2} \times \frac{a - x}{a + x} = \frac{2a\dot{x}}{a^2 - x^2},$$
$$\text{of } ax^n \text{ is } \frac{nax^{n-1}\dot{x}}{ax^n} = \frac{n\dot{x}}{x}.$$

24. By means of the fluxions of logarithms, are usually determined those of exponential quantities, that is, quantities which have their exponent a flowing or variable letter. These exponentials are of two kinds, namely, when the root is a constant quantity, as e^x, and when the root is variable as well as the exponent, as y^x.

25. In the first case, put the exponential, whose fluxion is to be found, equal to a single variable quantity z, namely, $z = e^x$; then take the logarithm of each, so shall $\log. z = x \times \log. e$; take the fluxions of these, so shall $\frac{\dot{z}}{z} = \dot{x} \times \log. e$, by the last article; hence $\dot{z} = z\dot{x} \times \log. e = e^x \dot{x} \times \log. e$, which is the fluxion of the proposed quantity e^x or z; and which therefore is equal to the said given quantity drawn into the fluxion of the exponent, and into the log. of the root.

Hence also, the fluxion of $(a + c)^{nx}$ is $(a + c)^{nx} \times n\dot{x} \times \log. (a + c)$.

26. In like manner, in the second case, put the given quantity $y^x = z$; then the logarithms give $\log. z = x \times \log. y$, and the fluxions give $\frac{\dot{z}}{z} = \dot{x} \times \log. y + x . \times \frac{\dot{y}}{y}$; hence $\dot{z} = z\dot{x} \times \log. y + \frac{zx\dot{y}}{y} =$ (by substituting y^x for z) $y^x\dot{x} \times \log. y + xy^{x-1}\dot{y}$, which is the fluxion of the proposed quantity y^x; and which therefore consist of two terms, of wh

the one is the fluxion of the given quantity considering the exponent as constant, and the other the fluxion of the same quantity considering the root as constant.

OF SECOND, THIRD, &c, FLUXIONS.

HAVING explained the manner of considering and determining the first fluxions of flowing or variable quantities; it remains now to consider those of the higher orders, as second, third, fourth, &c, fluxions.

27. If the rate or celerity with which any flowing quantity changes its magnitude, be constant, or the same at every position; then is the fluxion of it also constantly the same. But if the variation of magnitude be continually changing, either increasing or decreasing; then will there be a certain degree of fluxion peculiar to every point or position; and the rate of variation or change in the fluxion, is called the Fluxion of the Fluxion, or the Second Fluxion of the given fluent quantity. In like manner, the variation or fluxion of this second fluxion, is called the Third Fluxion of the first proposed fluent quantity; and so on.

These orders of fluxions are denoted by the same fluent letter with the corresponding number of points over it: namely, two points for the second fluxion, three points for the third fluxion, four points for the fourth fluxion, and so on. So, the different orders of the fluxion of x, are $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, &c; where each is the fluxion of the one next before it.

28. This description of the higher orders of fluxions may be illustrated by the figures exhibited in art. 8, page 288; where, if x denote the absciss AP, and y the ordinate PQ; and if the ordinate PQ or y flow along the absciss AP or x, with a uniform motion; then the fluxion of x, namely, $\dot{x} = \text{P}p$ or $\text{Q}r$, is a constant quantity, or $\ddot{x} = 0$, in all the figures. Also, in fig. 1, in which AQ is a right line, $\dot{y} = rq$, or the fluxion of PQ, is a constant quantity, or $\ddot{y} = 0$; for, the angle Q, = the angle A, being constant, Qr is to rq, or $\dot{x}$ to $\dot{y}$, in a constant ratio. But in the 2d fig. rq, or the fluxion of PQ, continually increases more and more; and in

in fig. 3 it continually decreases more and more, and therefore in both these cases y has a second fluxion, being positive in fig. 2, but negative in fig. 3. And so on, for the other orders of fluxions.

Thus if, for instance, the nature of the curve be such, that x^3 is every where equal to a^2y; then, taking the fluxions, it is $a^2\dot{y} = 3x^2\dot{x}$; and, considering $\dot{x}$ always as a constant quantity, and taking always the fluxions, the equations of the several orders of fluxions will be as below, viz.

the 1st fluxions $a^2\dot{y} = 3x^2\dot{x}$.

the 2d fluxions $a^2\ddot{y} = 6x\dot{x}^2$,

the 3d fluxions $a^2\dddot{y} = 6\dot{x}^3$,

the 4th fluxions $a^2\ddddot{y} = 0$,

and all the higher fluxions also $= 0$, or nothing.

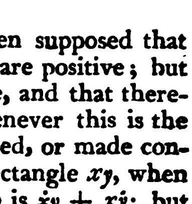
Also, the higher orders of fluxions are found in the same manner as the lower ones. Thus,

the first fluxion of y^3 is - - - - - - $3y^2\dot{y}$;

its 2d flux. or the flux. of $3y^2\dot{y}$, considered as the rectangle of $3y^2$, and $\dot{y}$, is - - - - - - - - $3y^2\ddot{y} + 6y\dot{y}^2$:

and the flux. of this again, or the 3d flux. of y^3, is - - - - - - - $3y^2\dddot{y} + 18y\dot{y}\ddot{y} + 6\dot{y}^3$.

29. In the foregoing articles, it has been supposed that the fluents increase, or that their fluxions are positive; but it often happens that some fluents decrease, and that therefore their fluxions are negative: and whenever this is the case, the sign of the fluxion must be changed, or made contrary to that of the fluent. So, of the rectangle xy, when both x and y increase together, the fluxion is $\dot{x}y + x\dot{y}$; but if one of them, as y, decrease, while the other, x, increases; then, the fluxion of y being $-\dot{y}$, the fluxion of xy will in that case be $\dot{x}y - x\dot{y}$. This may be illustrated by the annexed rectangle, APQR $= xy$, supposed to be generated by the motion of the line PQ from A towards C, and by the motion of the line RQ from B towards A: For, by the motion of PQ, from A towards C, the rectangle is increased, and its fluxion is $+ \dot{x}y$; but, by the motion of RQ, from B towards A, the rectangle is decreased, and the fluxion of the decrease is $x\dot{y}$; there-



fore,

fore, taking the fluxion of the decrease from that of the increase, the fluxion of the rectangle xy, when x increases and y decreases, is $\dot{x}y - x\dot{y}$.

30. We may now collect all the rules together, which have been demonstrated in the foregoing articles, for finding the fluxions of all sorts of quantities. And hence,

1st, *For the fluxion of Any Power of a flowing quantity.*—Multiply all together the exponent of the power, the fluxion of the root, and the power next less by 1 of the same root.

2d, *For the fluxion of the Rectangle of two quantities.*—Multiply each quantity by the fluxion of the other, and connect the two products together by their proper signs.

3d, *For the fluxion of the Continual Product of any number of flowing quantities.*—Multiply the fluxion of each quantity by the product of all the other quantities, and connect all the products together by their proper signs.

4th, *For the fluxion of a Fraction.*—From the fluxion of the numerator drawn into the denominator, subtract the fluxion of the denominator drawn into the numerator, and divide the result by the square of the denominator.

5th, *Or, the 2d, 3d, and 4th cases may be all included under one, and performed thus.*—Take the fluxion of the given expression as often as there are variable quantities in it, supposing first only one of them variable, and the rest constant; then another variable, and the rest constant; and so on, till they have all in their turns been singly supposed variable; and connect all these fluxions together with their own signs

6th, *For the Fluxion of a Logarithm.*—Divide the fluxion of the quantity by the quantity itself, and multiply the result by the modulus of the system of logarithms.

Note. The modulus of the hyperbolic logarithms is 1, and the modulus of the common logs, is - 0·43429448.

7th, *For the fluxion of an Exponential quantity, having the Root Constant.*—Multiply all together, the given quantity the fluxion of its exponent, and the hyp. log. of the root.

8th, *For the fluxion of an Exponential quantity having the Root Variable.*—To the fluxion of the given quantity, found by the 1st rule, as if the root only were variable, add the fluxion of the same quantity found by the 7th rule, as if the exponent only were variable; and the sum will be the fluxion for both of them variable.

Note. When the given quantity consists of several terms, find the fluxion of each term separately, and connect them all together with their proper signs.

31. PRACTICAL

31. PRACTICAL EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. The fluxion of axy is
2. The fluxion of $bxyz$ is
3. The fluxion of $cx \times (ax - cy)$ is
4. The fluxion of $x^m y^n$ is
5. The fluxion of $x^m y^n z^r$ is
6. The fluxion of $(x + y) \times (x - y)$ is
7. The fluxion of $2ax^2$ is
8. The fluxion of $2x^3$ is
9. The fluxion of $3x^4 y$ is
10. The fluxion of $4x^{\frac{2}{3}} y^4$ is
11. The fluxion of $ax^2 y - x^{\frac{1}{2}} y^3$ is
12. The fluxion of $4x^4 - x^2 y + 3byz$ is
13. The fluxion of $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$ is
14. The fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$ is
15. The fluxion of $\frac{1}{\sqrt[n]{x^m}}$ or $\frac{1}{x^{\frac{m}{n}}}$ or $x^{-\frac{m}{n}}$ is
16. The fluxion of $\sqrt{x}$ or $x^{\frac{1}{2}}$ is
17. The fluxion of $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$ is
18. The fluxion of $\sqrt[3]{x^2}$ or $x^{\frac{2}{3}}$ is
19. The fluxion of $\sqrt{x^3}$ or $x^{\frac{3}{2}}$ is
20. The fluxion of $\sqrt[4]{x^3}$ or $x^{\frac{3}{4}}$ is
21. The fluxion of $\sqrt[3]{x^4}$ or $x^{\frac{4}{3}}$ is
22. The fluxion of $\sqrt{(a^2 + x^2)}$ or $(a^2 + x^2)^{\frac{1}{2}}$ is
23. The fluxion of $\sqrt{(a^2 - x^2)}$ or $(a^2 - x^2)^{\frac{1}{2}}$ is
24. The fluxion of $\sqrt{(2rx - xx)}$ or $(2rx - xx)^{\frac{1}{2}}$ is
25. The fluxion of $\frac{1}{\sqrt{(a^2 - x^2)}}$ or $(a^2 - x^2)^{-\frac{1}{2}}$ is
26. The fluxion of $(ax - xx)^{\frac{1}{3}}$ is

27. The

27. The fluxion of $2x\sqrt{a^2 \pm x^2}$ is

28. The fluxion of $(a^2 - x^2)^{\frac{3}{2}}$ is

29. The fluxion of $\sqrt{xz}$ or $(xz)^{\frac{1}{2}}$ is

30. The fluxion of $\sqrt{xz - zz}$ or $(xz - zz)^{\frac{1}{2}}$ is

31. The fluxion of $-\dfrac{1}{a\sqrt{x}}$ or $-\frac{1}{a}x^{-\frac{1}{2}}$ is

32. The fluxion of $\dfrac{ax^3}{a+x}$ is

33. The fluxion of $\dfrac{x^m}{y^n}$ is

34. The fluxion of $\dfrac{xy}{z}$ is

35. The fluxion of $\dfrac{c}{xx}$ is

36. The fluxion of $\dfrac{3x}{a-x}$ is

37. The fluxion of $\dfrac{z}{x+z}$ is

38. The fluxion of $\dfrac{x^2}{z^2}$ is

39. The fluxion of $\dfrac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}$ is

40. The fluxion of $\dfrac{axy^2}{z}$ is

41. The fluxion of $\dfrac{3}{\sqrt{(x^2-y^2)}}$ is

42. The fluxion of the hyp. log. of ax is
43. The fluxion of the hyp. log. of $1 + x$ is
44. The fluxion of the hyp. log. of $1 - x$ is
45. The fluxion of the hyp. log. of x^2 is
46. The fluxion of the hyp. log. of $\sqrt{z}$ is
47. The fluxion of the hyp. log. of x^m is

48. The

48. The fluxion of the hyp. log. of $\frac{2}{x^2}$ is

49. The fluxion of the hyp. log. of $\frac{1+x}{1-x}$ is

50. The fluxion of the hyp. log. of $\frac{1-x}{1+x}$ is

51. The fluxion of c^x is

52. The fluxion of 10^x is

53. The fluxion of $(a+c)^x$ is

54. The fluxion of 100^{xy} is

55. The fluxion of x^2 is

56. The fluxion of y^{10x} is

57. The fluxion of x^x is

58. The fluxion of $(xy)^{xz}$ is

59. The fluxion of $x\dot{y}$ is

60. The fluxion of $\dot{x}\dot{y}^2$ is

61. The second fluxion of xy is

62. The second fluxion of xy, when $\dot{x}$ is constant, is

63. The second fluxion of x^n is

64. The third fluxion of x^n, when $\dot{x}$ is constant, is

65. The third fluxion of xy is

THE INVERSE METHOD, OR THE FINDING OF FLUENTS.

32. IT has been observed, that a Fluent, or Flowing Quantity, is the variable quantity which is considered as increasing or decreasing. Or, the fluent of a given fluxion, is such a quantity, that its fluxion, found according to the foregoing rules, shall be the same as the fluxion given or proposed.

33. It may further be observed, that Contemporary Fluents, or Contemporary Fluxions, are such as flow together, or for the same time.—When contemporary fluents are always equal, or in any constant ratio; then also are their fluxions respectively either equal, or in that same constant ratio. That is, if $x = y$, then is $\dot{x} = \dot{y}$; or if $x : y :: n : 1$, then is $\dot{x} : \dot{y} :: n : 1$; or if $x = ny$, then is $\dot{x} = n\dot{y}$.

34. It is easy to find the fluxions to all the given forms of fluents; but, on the contrary, it is difficult to find the fluents of many given fluxions; and indeed there are numberless cases in which this cannot at all be done, excepting by the quadrature and rectification of curve lines, or by logarithms, or by infinite series. For, it is only in certain particular forms and cases that the fluents of given fluxions can be found; there being no method of performing this universally, *a priori*, by a direct investigation, like finding the fluxion of a given fluent quantity. We can only therefore lay down a few rules for such forms of fluxions as we know, from the direct method, belong to such and such kinds of flowing quantities: and these rules, it is evident, must chiefly consist in performing such operations as are the reverse of those by which the fluxions are found of given fluent quantities. The principal cases of which are as follow.

35. *To find the Fluent of a Simple Fluxion; or of that in which there is no variable quantity, and only one fluxional quantity.*

This is done by barely substituting the variable or flowing quantity instead of its fluxion; being the result or reverse of the notation only.—Thus,

The fluent of $a\dot{x}$ is ax.
The fluent of $a\dot{y} + 2\dot{y}$ is $ay + 2y$.
The fluent of $\dot{\overline{\sqrt{a^2 + x^2}}}$ is $\sqrt{a^2 + x^2}$.

36. *When any Power of a flowing quantity is Multiplied by the Fluxion of the Root:*

Then, having substituted, as before, the flowing quantity, for its fluxion, divide the result by the new index of the power. Or, which is the same thing, take out, or divide by, the fluxion of the root; add 1 to the index of the power; and divide by the index so increased. Which is the reverse of the 1st rule for finding fluxions.

So, if the fluxion proposed be - - $3x^5\dot{x}$.
Leave out, or divide by, $\dot{x}$, then it is - $3x^5$;
add 1 to the index, and it is - - - $3x^6$;
divide by the index 6, and it is - - $\frac{3}{6}x^6$ or $\frac{1}{2}x^6$,
which is the fluent of the proposed fluxion $3x^5\dot{x}$.

In like manner,
The fluent of $2ax\dot{x}$ is ax^2.
The fluent of $3x^2\dot{x}$ is x^3.

The

The fluent of $4x^{\frac{1}{2}}\dot{x}$ is $\frac{8}{3}x^{\frac{3}{2}}$.

The fluent of $2y^{\frac{3}{4}}\dot{y}$ is $\frac{8}{7}y^{\frac{7}{4}}$.

The fluent of $az^{\frac{5}{6}}\dot{z}$ is $\frac{6}{11}az^{\frac{11}{6}}$.

The fluent of $x^{\frac{1}{2}}\dot{x} + 3y^{\frac{2}{3}}\dot{y}$ is $\frac{2}{3}x^{\frac{3}{2}} + \frac{9}{5}y^{\frac{5}{3}}$.

The fluent of $x^{n-1}\dot{x}$ is $\frac{1}{n}x^{n}$.

The fluent of $ny^{n-1}\dot{y}$ is

The fluent of $\frac{\dot{z}}{z^2}$, or $z^{-2}\dot{z}$ is

The fluent of $\frac{a\dot{y}}{y^n}$ is

The fluent of $(a + x)^4\dot{x}$ is

The fluent of $(a^4 + y^4)y^3\dot{y}$ is

The fluent of $(a^3 + z^3)^4 z^2\dot{z}$ is

The fluent of $(a^n + x^n)^m x^{n-1}\dot{x}$ is

The fluent of $(a^2 + y^2)^3 y\dot{y}$ is

The fluent of $\frac{z\dot{z}}{\sqrt{(a^2+z^2)}}$ is

The fluent of $\frac{\dot{x}}{\sqrt{(a-x)}}$ is

37. *When the Root under a Vinculum is a Compound Quantity; and the Index of the part or factor Without the Vinculum, increased by* 1, *is some Multiple of that Under the Vinculum :*

Put a single variable letter for the compound root; and substitute its powers and fluxion instead of those of the same value, in the given quantity; so will it be reduced to a simpler form, to which the preceding rule can then be applied.

Thus, if the given fluxion be $\dot{F} = (a^2 + x^2)^{\frac{2}{3}}x^3\dot{x}$, where 3, the index of the quantity without the vinculum, increased by 1, making 4, which is just the double of 2, the exponent of x^2 within the vinculum : therefore, putting $z = a^2 + x^2$, thence $x^2 = z - a^2$, the fluxion of which is $2x\dot{x} = \dot{z}$; hence then $x^3\dot{x} = \frac{1}{2}x^2\dot{z} = \frac{1}{2}\dot{z}(z - a^2)$, and the given fluxion $\dot{F}$, or $(a^2 + x^2)^{\frac{2}{3}}x^3\dot{x}$, is $= \frac{1}{2}z^{\frac{2}{3}}\dot{z}(z - a^2)$ or $= \frac{1}{2}z^{\frac{5}{3}}\dot{z} - \frac{1}{2}a^2z^{\frac{2}{3}}\dot{z}$; and hence the fluent F is $= \frac{3}{16}z^{\frac{8}{3}} - \frac{3}{10}a^2z^{\frac{5}{3}} = 3z^{\frac{5}{3}}(\frac{1}{16}z - \frac{1}{10}a^2)$. Or, by substituting the value of z instead of it, the same fluent is $3(a^2 + x^2)^{\frac{5}{3}} \times (\frac{1}{16}x^2 - \frac{3}{80}a^2)$, or $\frac{3}{16}(a^2 + x^2) \times (x^2 - \frac{3}{5}a^2)$.

In

In like manner for the following examples.

To find the fluent of $\sqrt{a + cx} \times x^3\dot{x}$.

To find the fluent of $(a + cx)^{\frac{3}{2}}x^2\dot{x}$.

To find the fluent of $(a + cx^2)^{\frac{1}{3}} \times dx^3\dot{x}$.

To find the fluent of $\frac{cz\dot{z}}{\sqrt{a + z}}$ or $(a + z)^{\frac{1}{2}}cz\dot{z}$.

To find the fluent of $\frac{cz^{3n-1}\dot{z}}{\sqrt{a + z^n}}$ or $(a + z^n)^{-\frac{1}{2}}cz^{3n-1}\dot{z}$.

To find the fluent of $\frac{\dot{z}\sqrt{a^2 + z^2}}{z^6}$ or $(a^2 + z^2)^{\frac{1}{2}}z^{-6}\dot{z}$.

To find the fluent of $\frac{\dot{x}\sqrt{a - x^n}}{x^{\frac{7}{2}n-1}}$ or $(a - x^n)^{\frac{1}{2}}x^{\frac{7}{2}n-1}\dot{x}$.

38. *When there are several Terms, involving Two or more Variable Quantities, having the Fluxion of each Multiplied by the other Quantity or Quantities:*

Take the fluent of each term, as if there were only one variable quantity in it, namely, that whose fluxion is contained in it, supposing all the others to be constant in that term; then, if the fluents of all the terms, so found, be the very same quantity in all of them, that quantity will be the fluent of the whole. Which is the reverse of the 5th rule for finding fluxions: Thus, if the given fluxion be $\dot{x}y + x\dot{y}$, then the fluent of $\dot{x}y$ is xy, supposing y constant: and the fluent of $x\dot{y}$ is also xy, supposing x constant: therefore xy is the required fluent of the given fluxion $\dot{x}y + x\dot{y}$.

In like manner,

The fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is xyz.

The fluent of $2xy\dot{x} + x^2\dot{y}$ is x^2y.

The fluent of $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^2 + 2x^{\frac{1}{2}}y\dot{y}$ is

The fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$ is

The fluent of $\frac{2ax\dot{x}y^{\frac{1}{2}} - \frac{1}{2}ax^2y^{-\frac{1}{2}}\dot{y}}{y}$ $\frac{2ax\dot{x}}{\sqrt{y}} - \frac{ax^2\dot{y}}{2y\sqrt{y}}$ is

39. When

39. *When the given Fluxional Expression is in this Form* $\frac{\dot{x}y - x\dot{y}}{y^2}$, *namely, a Fraction, including Two Quantities, being the Fluxion of the former of them drawn into the latter, minus the Fluxion of the latter drawn into the former, and divided by the Square of the latter:*

Then, the fluent is the fraction $\frac{x}{y}$, or the former quantity divided by the latter. That is,

The fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ is $\frac{x}{y}$. And, in like manner,

The fluent of $\frac{2x\dot{x}y^2 - 2x^2y\dot{y}}{y^4}$ is $\frac{x^2}{y^2}$.

Though, indeed, the examples of this case may be performed by the foregoing one. Thus, the given fluxion - - $\frac{\dot{x}y - x\dot{y}}{y^2}$ reduces to $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$, or $\frac{\dot{x}}{y} - x\dot{y}y^{-2}$; of which,

the fluent of $\frac{\dot{x}}{y}$ is $\frac{x}{y}$ supposing y constant; and

the fluent of $- x\dot{y}y^{-2}$ is also xy^{-1} or $\frac{x}{y}$, when x is constant;

therefore, by that case, $\frac{x}{y}$ is the fluent of the whole $\frac{\dot{x}y - x\dot{y}}{y^2}$.

40. *When the Fluxion of a Quantity is Divided by the Quantity itself:*

Then the fluent is equal to the hyperbolic logarithm of that quantity; or, which is the same thing, the fluent is equal to 2·30258509 multiplied by the common logarithm of the same quantity.

So, the fluent of $\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$, is the hyp. log. of x.

The fluent of $\frac{2\dot{x}}{x}$ is $2 \times$ hyp. log. of x, or = hyp. log. x^2.

The fluent of $\frac{a\dot{x}}{x}$, is $a \times$ hyp. log. x, or = hyp. log. of x^a.

The fluent of $\frac{\dot{x}}{a + x}$, is

The fluent of $\frac{3x^2\dot{x}}{a + x^3}$, is

41. *Many*

41. *Many fluents may be found by the Direct Method thus:*

Take the fluxion again of the given fluxion, or the second fluxion of the fluent sought; into which substitute $\frac{\dot{x}^2}{x}$ for $\ddot{x}$, $\frac{\dot{y}^2}{y}$ for $\ddot{y}$, &c; that is, make $x, \dot{x}, \ddot{x}$, as also $y, \dot{y}, \ddot{y}$, &c, to be in continual proportion, or so that $x : \dot{x} :: \dot{x} : \ddot{x}$, and $y : \dot{y} :: \dot{y} : \ddot{y}$, &c; then divide the square of the given fluxional expression by the second fluxion, just found, and the quotient will be the fluent required in many cases.

Or the same rule may be otherwise delivered thus:

In the given fluxion $\dot{F}$, write x for $\dot{x}$, y for $\dot{y}$, &c, and call the result G, taking also the fluxion of this quantity, $\dot{G}$; then make $\dot{G} : \dot{F} :: G : F$; so shall the fourth proportional F be the fluent sought in many cases.

It may be proved if this be the true fluent, by taking the fluxion of it again, which, if it agree with the proposed fluxion, will show that the fluent is right; otherwise, it is wrong.

EXAMPLES.

EXAM. 1. Let it be required to find the fluent of $nx^{n-1}\dot{x}$.

Here $\dot{F} = nx^{n-1}\dot{x}$. Write x for $\dot{x}$, then $nx^{n-1}x$ or $nx^n = G$; the fluxion of this is $\dot{G} = n^2x^{n-1}\dot{x}$; therefore $\dot{G} : \dot{F} :: G : F$, becomes $n^2x^{n-1}\dot{x} : nx^{n-1}\dot{x} :: nx^n : x^n = F$, the fluent sought.

EXAM. 2. To find the fluent of $\dot{x}y + x\dot{y}$.

Here $\dot{F} = \dot{x}y + x\dot{y}$; then, writing x for $\dot{x}$, and y for $\dot{y}$, it is $xy + xy$ or $2xy = G$; hence $\dot{G} = 2\dot{x}y + 2x\dot{y}$; then $\dot{G} : \dot{F} :: G : F$, becomes $2\dot{x}y + 2x\dot{y} : \dot{x}y + x\dot{y} :: 2xy : xy = F$, the fluent sought.

42. *To find Fluents by means of a Table of Forms of Fluxions and Fluents.*

In the following Table are contained the most usual forms of fluxions that occur in the practical solution of problems, with their corresponding fluents set opposite to them; by means of which, namely, by comparing any proposed fluxion with the corresponding form in the table, the fluent of it will be found.

Forms.

Forms.	*Fluxions.*	*Fluents.*
I	$x^{n-1}\dot{x}$	$\frac{x^n}{n}$ or $\frac{1}{n}x^n$
II	$(a \pm x^n)^{m-1}x^{n-1}\dot{x}$	$\pm\frac{1}{mn}(a \pm x^n)^m$
III	$\frac{x^{mn-1}\dot{x}}{(a \pm x^n)^{m+1}}$	$\frac{1}{mna} \times \frac{x^{mn}}{(a \pm x^n)^m}$
IV	$\frac{(a \pm x^n)^{m-1}\dot{x}}{x^{mn+1}}$	$\frac{-1}{mna} \times \frac{(a \pm x^n)^m}{x^{mn}}$
V	$(my\dot{x}+nx\dot{y}) \times x^{m-1}y^{n-1}$, or $(\frac{m\dot{x}}{x}+\frac{n\dot{y}}{y})x^m y^n$	$x^m y^n$
VI	$mx^{m-1}\dot{x}y^n z^r + nx^m y^{n-1}\dot{y}z^r + rx^m y^n z^{r-1}\dot{z}$, or $(m\dot{x}yz + nx\dot{y}z + rxy\dot{z})x^{m-1}y^{n-1}z^{r-1}$, or $(\frac{m\dot{x}}{x}+\frac{n\dot{y}}{y}+\frac{r\dot{z}}{z})x^m y^n z^r$,	$x^m y^n z^r$
VII	$\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$	log. of x
VIII	$\frac{x^{n-1}\dot{x}}{a \pm x^n}$	$\pm\frac{1}{n}$ log. of $a \pm x^n$
IX	$\frac{x^{-1}\dot{x}}{a \pm x^n}$	$\frac{1}{na}$ log of $\frac{x^n}{a \pm x^n}$
X	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a - x^n}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\sqrt{a}+\sqrt{x^n}}{\sqrt{a}-\sqrt{x^n}}$
XI	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a+x^n}$	$\frac{2}{n\sqrt{a}} \times$ arc to tan. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}} \times$ arc to cosine $\frac{a-x^n}{a+x^n}$
XII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{\pm a + x^n}}$	$\frac{2}{n}$ log. of $\sqrt{x^n} + \sqrt{\pm a + x^n}$

Form-

Forms.	*Fluxions.*	*Fluents.*
XIII	$\dfrac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{a-x^n}}$	$\dfrac{2}{h}\times$ arc to sin. $\sqrt{\dfrac{x^n}{a}}$, or $\dfrac{1}{n}\times$ arc to vers. $\dfrac{2x^n}{a}$
XIV	$\dfrac{x^{-1}\dot{x}}{\sqrt{a\pm x^n}}$	$\dfrac{1}{n\sqrt{a}}$ log of $\dfrac{\pm\sqrt{a\pm x^n}\mp\sqrt{a}}{\sqrt{a\pm x^n}+\sqrt{a}}$
XV	$\dfrac{x^{-1}\dot{x}}{\sqrt{-a+x^n}}$	$\dfrac{2}{n\sqrt{a}}\times$ arc to secant $\sqrt{\dfrac{x^n}{a}}$, or $\dfrac{1}{n\sqrt{a}}\times$ arc to cosin. $\dfrac{2a-x^n}{x^n}$
XVI	$\dot{x}\sqrt{dx-x^2}$	$\dfrac{1}{2}$ circ. seg. to diam. d & vers. x
XVII	$c^{nx}\dot{x}$	$\dfrac{c^{nx}}{n \log. c}$
XVIII	$\dot{x}y^x \log. y + xy^{x-1}\dot{y}$	y^x

Note. The logarithms, in the above forms, are the hyperbolic ones, which are found by multiplying the common logarithms by 2·302585092994. And the arcs, whose sine, or tangent, &c, are mentioned, have the radius 1, and are those in the common tables of sines, tangents and secants. Also, the numbers m, n, &c, are to be some real quantities, as the forms fail when $m = 0$, or $n = 0$, &c.

The Use of the foregoing Table of Forms of Fluxions and Fluents.

43. In using the foregoing table, it is to be observed, that the first column serves only to show the number of the form; in the second column are the several forms of fluxions, which are of different kinds or classes; and in the third or last column, are the corresponding fluents.

The method of using the table, is this. Having any fluxion given, to find its fluent: First, Compare the given fluxion with the several forms of fluxions in the second column of the table, till one of the forms be found that agrees with it; which is done by comparing the terms of the given fluxion with the like parts of the tabular fluxion, namely, the radical quantity of the one, with that of the other; and the

the exponents of the variable quantities of each, both within and without the vinculum; all which, being found to agree or correspond, will give the particular values of the general quantities in the tabular form: then substitute these particular values in the general or tabular form of the fluent, and the result will be the particular fluent of the given fluxion; after it is multiplied by any co-efficient the proposed fluxion may have.

EXAMPLES.

EXAM. 1. To find the fluent of the fluxion $3x^{\frac{5}{3}}\dot{x}$.

This is found to agree with the first form. And, by comparing the fluxions, it appears that $x = x$, and $n - 1 = \frac{5}{3}$, or $n = \frac{8}{3}$; which being substituted in the tabular fluent, or $\frac{1}{n}x^n$, gives, after multiplying by 3 the co-efficient, $3 \times \frac{3}{8}x^{\frac{8}{3}}$, or $\frac{9}{8}x^{\frac{8}{3}}$, for the fluent sought.

EXAM. 2. To find the fluent of $5x^2\dot{x}\sqrt{c^3 - x^3}$, or $5x^2\dot{x}(c^3 - x^3)^{\frac{1}{2}}$.

This fluxion, it appears, belongs to the 2d tabular form: for $a = c^3$, and $-x^n = -x^3$, and $n = 3$ under the vinculum, also $m - 1 = \frac{1}{2}$, or $m = \frac{3}{2}$, and the exponent $^{n-1}$ of x^{n-1} without the vinculum, by using 3 for n, is $n - 1 = 2$, which agrees with x^2 in the given fluxion: so that all the parts of the form are found to correspond. Then, substituting these values into the general fluent, $\frac{1}{-mn}(a - x^n)^m$,

it becomes $-\frac{5}{3} \times \frac{2}{3}(c^3 - x^3)^{\frac{3}{2}} = -\frac{10}{9}(c^3 - x^3)^{\frac{3}{2}}$.

EXAM. 3. To find the fluent of $\frac{x^2\dot{x}}{1 + x^3}$.

This is found to agree with the 8th form; where $\pm x^n = + x^3$ in the denominator, or $n = 3$; and the numerator x^{n-1} then becomes x^2, which agrees with the numerator in the given fluxion; also $a = 1$. Hence then, by substituting in the general or tabular fluent, $\frac{1}{n}$ log. of $a + x^n$, it becomes $\frac{1}{3}$ log. $1 + x^3$.

EXAM. 4. To find the fluent of $ax^4\dot{x}$.

EXAM. 5. To find the fluent of $2(10 + x^2)^{\frac{2}{3}}x\dot{x}$.

EXAM. 6. To find the fluent of $\frac{a\dot{x}}{(c^2 + x^2)^{\frac{3}{2}}}$.

EXAM. 7. To find the fluent of $\frac{3x^2\dot{x}}{(a - x^3)^4}$.

EXAM. 8.

EXAM. 8. To find the fluent of $\frac{c^2 - x^2}{x^5}\dot{x}$.

EXAM. 9. To find the fluent of $\frac{1 + 3x}{2x^4}\dot{x}$.

EXAM. 10. To find the fluent of $(\frac{3\dot{x}}{x} + \frac{2\dot{y}}{y})\, x^3y^2$.

EXAM. 11. To find the fluent of $(\frac{\dot{x}}{x} + \frac{\dot{y}}{3y})xy^{\frac{1}{3}}$.

EXAM. 12. To find the fluent of $\frac{3\dot{x}}{ax}$ or $\frac{3}{a}x^{-1}\dot{x}$.

EXAM. 13. To find the fluent of $\frac{a\dot{x}}{3 - 2x}$.

EXAM. 14. To find the fluent of $\frac{3\dot{x}}{2x - x^2}$ or $\frac{3x^{-1}\dot{x}}{2 - \dot{x}}$.

EXAM. 15. To find the fluent of $\frac{2\dot{x}}{x - 3x^3}$ or $\frac{2x^{-1}3\dot{x}^2}{1 - 3\dot{x}^2}$

EXAM. 16. To find the fluent of $\frac{3x\dot{x}}{1 - x^4}$.

EXAM. 17. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2 - x^5}$?

EXAM. 18. To find the fluent of $\frac{2x\dot{x}}{1 + x^4}$.

EXAM. 19. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2 + x^5}$.

EXAM. 20. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1 + x^4}}$.

EXAM. 21. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^2 - 4}}$.

EXAM. 22. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1 - x^4}}$.

EXAM. 23. To find the fluent of $\frac{a\dot{x}}{\sqrt{4 - x^2}}$.

EXAM. 24. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{1 - x^2}}$.

EXAM. 25. To find the fluent of $\frac{a\dot{x}}{\sqrt{ax^2 + x^{\frac{1}{2}}}}$.

EXAM. 26. To find the fluent of $\frac{2x^{-}\dot{x}}{\sqrt{x^2 - 1}}$.

EXAM. 27.

the fluent of $\dfrac{a\dot{x}}{\sqrt{x^{\frac{1}{2}} - ax^2}}$.

of $2\dot{x}\sqrt{2x - x^2}$.

of $a^x\dot{x}$.

ent of $3a^{2x}\dot{x}$.

uent of $3z^x\dot{x}$ log. $z + 3xz^{x-1}\dot{z}$.

the fluent of $(1 + x^3)^{\frac{1}{3}} x\dot{x}$.

find the fluent of $(2 + x^4)\, x^{\frac{3}{2}}\dot{x}$.

To find the fluent of $x^2\dot{x}\sqrt{a^2 + x^2}$.

To find Fluents by Infinite Series.

44. When a given fluxion, whose fluent is required, is so complex, that it cannot be made to agree with any of the forms in the foregoing table of cases, nor made out from the general rules before given; recourse may then be had to the method of infinite series; which is thus performed:

Expand the radical or fraction, in the given fluxion, into an infinite series of simple terms, by the methods given for that purpose in books of algebra; viz. either by division or extraction of roots, or by the binomial theorem, &c; and multiply every term by the fluxional letter, and by such simple variable factor as the given fluxional expression may contain. Then take the fluent of each term separately, by the foregoing rules, connecting them all together by their proper signs; and the series will be the fluent sought, after it is multiplied by any constant factor or co-efficient which may be contained in the given fluxional expression.

45. It is to be noted however, that the quantities must be so arranged, as that the series produced may be a converging one, rather than diverging: and this is effected by placing the greater terms foremost in the given fluxion. When these are known or constant quantities, the infinite series will be an ascending one; that is, the powers of the variable quantity will ascend or increase; but if the variable quantity be set foremost, the infinite series produced will be a descending one, or the powers of that quan[illegible] de-crease always more and more in the succeedin[illegible] crease in the denominators of them, which is t[illegible]

For example, to find the fluent of $\frac{1-x}{1+x-x^2}\dot{x}$.

Here, by dividing the numerator by the denominator, the proposed fluxion becomes $\dot{x} - 2x\dot{x} + 3x^2\dot{x} - 5x^3\dot{x} + 8x^4\dot{x} - \&c$; then the fluents of all the terms being taken, give - - - $x - x^2 + x^3 - \frac{5}{4}x^4 + \frac{8}{5}x^5 - \&c$, for the fluent sought.

Again, to find the fluent of $\dot{x}\sqrt{1-x^2}$.

Here, by extracting the root, or expanding the radical quantity $\sqrt{1-x^2}$, the given fluxion becomes - - - - $\dot{x} - \frac{1}{2}x^2\dot{x} - \frac{1}{8}x^4\dot{x} - \frac{1}{16}x^6\dot{x} - \&c$. Then the fluents of all the terms, being taken, give $x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \&c$, for the fluent sought.

OTHER EXAMPLES.

EXAM. 1. To find the fluent of $\frac{bx\dot{x}}{a-x}$ both in an ascending and descending series.

EXAM. 2. To find the fluent of $\frac{b\dot{x}}{a+x}$ in both series.

EXAM. 3. To find the fluent of $\frac{3\dot{x}}{(a+x)^2}$.

EXAM. 4. To find the fluent of $\frac{1-x^2+2x^4}{1+x-x^2}\dot{x}$.

EXAM. 5. Given $\dot{z} = \frac{b\dot{x}}{a^2+x^2}$, to find z.

EXAM. 6. Given $\dot{z} = \frac{a^2+x^2}{a+x}\dot{x}$ to find z.

EXAM. 7. Given $\dot{z} = 3\dot{x}\sqrt{a+x}$, to find z.

EXAM. 8. Given $\dot{z} = 2\dot{x}\sqrt{a^2+x^2}$, to find z.

EXAM. 9. Given $\dot{z} = 4\dot{x}\sqrt{a^2-x^2}$, to find z.

EXAM. 10. Given $\dot{z} = \frac{5a\dot{x}}{\sqrt{x^2-a^2}}$, to find z.

EXAM. 11. Given $\dot{z} = 2\dot{x}\sqrt[3]{a^3-x^3}$, to find z.

EXAM. 12. Given $\dot{z} = \frac{3a\dot{x}}{\sqrt{ax-xx}}$, to find z.

EXAM. 13. Given $\dot{z} = 2\dot{x}\sqrt[3]{x^3+x^4+x^5}$, to find z.

EXAM. 14. Given $\dot{z} = 5\dot{x}\sqrt{ax-xx}$, to find z,

To

To Correct the Fluent of any Given Fluxion.

46. The fluxion found from a given fluent, is always perfect and complete; but the fluent found from a given fluxion is not always so; as it often wants a correction, to make it contemporaneous with that required by the problem under consideration, &c: for, the fluent of any given fluxion, as $\dot{x}$ may be either x, which is found by the rule, or it may be $x + c$, or $x - c$, that is x plus or minus some constant quantity c; because both x and $x \pm c$ have the same fluxion $\dot{x}$, and the finding of the constant quantity c, to be added or subtracted with the fluent as found by the foregoing rules, is called *correcting* the fluent.

Now this correction is to be determined from the nature of the problem in hand, by which we come to know the relation which the fluent quantities have to each other at some certain point or time. Reduce, therefore, the general fluential equation, supposed to be found by the foregoing rules, to that point or time; then if the equation be true, it is correct; but if not, it wants a correction; and the quantity of the correction, is the difference between the two general sides of the equation when reduced to that particular point. Hence the general rule for the correction is this:

Connect the constant, but indeterminate, quantity c, with one side of the fluential equation, as determined by the foregoing rules; then, in this equation, substitute for the variable quantities, such values as they are known to have at any particular state, place, or time; and then, from that particular state of the equation, find the value of c, the constant quantity of the correction.

EXAMPLES.

47. EXAM. 1. To find the correct fluent of $\dot{z} = ax^3\dot{x}$.

The general fluent is $z = ax^4$, or $z = ax^4 + c$, taking in the correction c.

Now, if it be known that z and x begin together, or that z is $= 0$, when $x = 0$; then writing 0 for both x and z, the general equation becomes $0 = 0 + c$, or $= c$; so that, the value of c being 0, the correct fluents are $z = ax^4$.

But if z be $= 0$, when x is $= b$, any known quantity; then substituting 0 for z, and b for x, in the general equation, it becomes $0 = ab^4 + c$, and hence we find $c = - ab^4$; which being written for c in the general fluential equation, it becomes $z = ax^4 - ab^4$, for the correct fluents.

Or,

Or, if it be known that z is $=$ some quantity d, when x is $=$ some other quantity as b; then substituting d for z, and b for x, in the general fluential equation $z = ax^4 + c$, it becomes $d = ab^4 + c$; and hence is deduced the value of the correction, namely, $c = d - ab^4$; consequently, writing this value for c in the general equation, it becomes - - - $z = ax^4 - ab^4 + d$, for the correct equation of the fluents in this case.

48. And hence arises another easy and general way of correcting the fluents, which is this: In the general equation of the fluents, write the particular values of the quantities which they are known to have at any certain time or position; then subtract the sides of the resulting particular equation from the corresponding sides of the general one, and the remainders will give the correct equation of the fluents sought.

So, the general equation being $z = ax^4$;
write d for z, and b for x, then $d = ab^4$;
hence, by subtraction, - $z - d = ax^4 - ab^4$,
or $z = ax^4 - ab^4 + d$, the correct fluents as before.

EXAM. 2. To find the correct fluents of $\dot{z} = 5x\dot{x}$; z being $= 0$ when x is $= a$.

EXAM. 3. To find the correct fluents of $\dot{z} = 3\dot{x}\sqrt{a + x}$; z and x being $= 0$ at the same time.

EXAM. 4. To find the correct fluent of $\dot{z} = \dfrac{2a\dot{x}}{a + x}$; supposing z and x to begin to flow together, or to be each $= 0$ at the same time.

EXAM. 5. To find the correct fluents of $\dot{z} = \dfrac{2\dot{x}}{a^2 + x^2}$; supposing z and x to begin together.

OF MAXIMA AND MINIMA; OR, THE GREATEST AND LEAST MAGNITUDE OF VARIABLE OR FLOWING QUANTITIES.

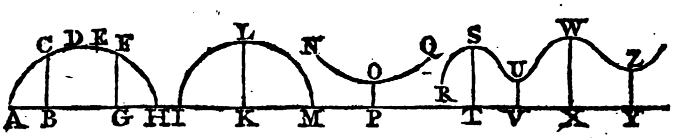
49. MAXIMUM, denotes the greatest state or quantity attainable in any given case, or the greatest value of a variable quantity: by which it stands opposed to Minimum, which is the least possible quantity in any case.

Thus,

Thus, the expression or sum $a^2 + bx$, evidently increases as x, or the term bx, increases; therefore the given expression will be the greatest, or a maximum, when x is the greatest, or infinite: and the same expression will be a minimum, or the least, when x is the least, or nothing.

Again, in the algebraic expression $a^2 - bx$, where a and b denote constant or invariable quantities, and x a flowing or variable one. Now, it is evident that the value of this remainder or difference, $a^2 - bx$, will increase, as the term bx, or as x, decreases; therefore the former will be the greatest, when the latter is the smallest; that is $a^2 - bx$ is a maximum, when x is the least, or nothing at all; and the difference is the least, when x is the greatest.

50. Some variable quantities increase continually; and so have no maximum, but what is infinite. Others again decrease continually; and so have no minimum, but what is of no magnitude, or nothing. But, on the other hand, some variable quantities increase only to a certain finite magnitude, called their Maximum, or greatest state, and after that they decrease again. While others decrease to a certain finite magnitude, called their Minimum, or least state, and afterwards increase again. And lastly, some quantities have several maxima and minima.



Thus, for example, the ordinate BC of the parabola, or such-like curve, flowing along the axis AB from the vertex A, continually increases, and has no limit or maximum. And the ordinate GF of the curve EFH, flowing from E towards H, continually decreases to nothing when it arrives at the point H. But in the circle ILM, the ordinate only increases to a certain magnitude, namely, the radius, when it arrives at the middle as at KL, which is its maximum; and after that it decreases again to nothing, at the point M. And in the curve NOQ, the ordinate decreases only to the position OP, where it is least, or a minimum; and after that it continually increases towards Q. But in the curve RSU &c, the ordinates have several maxima, as ST, WX, and several minima, as VU, YZ, &c.

51. Now,

51. Now, because the fluxion of a variable quantity, is the rate of its increase or decrease; and because the maximum or minimum of a quantity neither increases nor decreases, at those points or states; therefore such maximum or minimum has no fluxion, or the fluxion is then equal to nothing. From which we have the following rule.

To find the Maximum or Minimum.

52. From the nature of the question or problem, find an algebraical expression for the value, or general state, of the quantity whose maximum or minimum is required; then take the fluxion of that expression, and put it equal to nothing; from which equation, by dividing by, or leaving out, the fluxional letter and other common quantities, and performing other proper reductions, as in common algebra, the value of the unknown quantity will be obtained, determining the point of the maximum or minimum.

So, if it be required to find the maximum state of the compound expression $100x - 5x^2 \pm c$, or the value of x when $100x - 5x^2 \pm c$ is a maximum. The fluxion of this expression is $100\dot{x} - 10x\dot{x} = 0$; which being made $= 0$, and divided by $10\dot{x}$, the equation is $10 - x = 0$; and hence $x = 10$. That is, the value of x is 10, when the expression $100x - 5x^2 \pm c$ is the greatest. As is easily tried: for if 10 be substituted for x in that expression, it becomes $\pm c + 500$: but if, for x, there be substituted any other number, whether greater or less than 10, that expression will always be found to be less than $\pm c + 500$, which is therefore its greatest possible value, or its maximum.

53. It is evident, that if a maximum or minimum be any way compounded with, or operated on, by a given constant quantity, the result will still be a maximum or minimum. That is, if a maximum or minimum be increased, or decreased, or multiplied, or divided, by a given quantity, or any given power or root of it be taken; the result will still be a maximum or minimum. Thus, if x be a maximum or minimum, then also is $x + a$, or $x - a$, or ax, or $\frac{x}{a}$, or x^2, or $\sqrt[2]{x}$, still a maximum or minimum. Also, the logarithm of the same will be a maximum or a minimum. And therefore, if any proposed maximum or minimum can be made simpler by performing any of these operations, it is better to do so, before the expression is put into fluxions.

54. When

54. When the expression for a maximum or minimum contains several variable letters or quantities; take the fluxion of it as often as there are variable letters; supposing first one of them only to flow, and the rest to be constant; then another only to flow, and the rest constant; and so on for all of them: then putting each of these fluxions $= 0$, there will be as many equations as unknown letters, from which these may be all determined. For the fluxion of the expression must be equal to nothing in each of these cases; otherwise the expression might become greater or less, without altering the values of the other letters, which are considered as constant.

So, if it be required to find the values of x and y when $4x^2 - xy + 2y$ is a minimum. Then we have,

First, - $8x\dot{x} - \dot{x}y = 0$, and $8x - y = 0$, or $y = 8x$.
Secondly, $2\dot{y} - x\dot{y} = 0$, and $2 - x = 0$, or $x = 2$.
And hence y or $8x = 16$.

55. *To find whether a proposed quantity admits of a Maximum or a Minimum.*

Every algebraic expression does not admit of a maximum or minimum, properly so called; for it may either increase continually to infinity, or decrease continually to nothing; and in both these cases there is neither a proper maximum nor minimum; for the true maximum is that finite value to which an expression increases, and after which it decreases again: and the minimum is that finite value to which the expression decreases and after that it increases again. Therefore, when the expression admits of a maximum, its fluxion is positive before the point, and negative after it; but when it admits of a minimum, its fluxion is negative before, and positive after it. Hence then, taking the fluxion of the expression a little before the fluxion is equal to nothing, and again a little after the same; if the former fluxion be positive, and the latter negative, the middle state is a maximum; but if the former fluxion be negative, and the latter positive, the middle state is minimum.

So, if we would find the quantity $ax - x^2$ a maximum or minimum; make its fluxion equal to nothing, that is, - - $a\dot{x} - 2x\dot{x} = 0$, or $(a - 2x)\dot{x} = 0$; dividing by $\dot{x}$, gives $a - 2x = 0$, or $x = \frac{1}{2}a$ at that state. Now, if in the fluxion $(a - 2x)\dot{x}$, the value of x be taken rather less than its true value, $\frac{1}{2}a$, that fluxion will evidently be positive; but if x be taken somewhat greater than $\frac{1}{2}a$ the value of $a - 2x$, and consequently of the fluxion, is as evidently negative. Therefore, the fluxion of $ax - x^2$ being positive before, and negative

gative after the state when its fluxion is $= 0$, it follows that at this state the expression is not a minimum, but a maximum.

Again, taking the expression $x^3 - ax^2$, its fluxion $3x^2\dot{x} - 2ax\dot{x} = (3x - 2a)x\dot{x} = 0$; this divided by $x\dot{x}$ gives $3x - 2a = 0$, and $x = \frac{2}{3}a$, its true value when the fluxion of $x^3 - ax^2$ is equal to nothing. But now to know whether the given expression be a maximum or a minimum at that time, take x a little less than $\frac{2}{3}a$ in the value of the fluxion $(3x - 2a)x\dot{x}$, and this will evidently be negative; and again, taking x a little more than $\frac{2}{3}a$, the value of $3x - 2a$, or of the fluxion, is as evidently positive. Therefore the fluxion of $x^3 - ax^2$ being negative before that fluxion is $= 0$, and positive after it, it follows that in this state the quantity $x^3 - ax^2$ admits of a minimum, but not of a maximum.

56. Some Examples for Practice.

Exam. 1. To divide a line, or any other given quantity a, into two parts, so that their rectangle or product may be the greatest possible.

Exam. 2. To divide the given quantity a into two parts such, that the product of the m power of one, by the n power of the other, may be a maximum.

Exam. 3. To divide the given quantity a into three parts such, that the continual product of them all may be a maximum.

Exam. 4. To divide the given quantity a into three parts such, that the continual product of the 1st, the square of the 2d, and the cube of the 3d, may be a maximum.

Exam. 5. To determine a fraction such, that the difference between its m power and n power shall be the greatest possible.

Exam. 6. To divide the number 80 into two such parts, x and y, that $2x^2 + xy + 3y^2$ may be a minimum.

Exam. 7. To find the greatest rectangle that can be inscribed in a given right-angled triangle.

Exam. 8. To find the greatest rectangle that can be inscribed in the quadrant of a given circle.

Exam. 9. To find the least right-angled triangle that can circumscribe the quadrant of a given circle.

Exam. 10. To find the greatest rectangle inscribed in, and the least isosceles triangle circumscribed about, a given semi-ellipse.

EXAM. 11. To determine the same for a given parabola.

EXAM. 12. To determine the same for a given hyperbola.

EXAM. 13. To inscribe the greatest cylinder in a given cone; or to cut the greatest cylinder out of a given cone.

EXAM. 14. To determine the dimensions of a rectangular cistern, capable of containing a given quantity a of water, so as to be lined with lead at the least possible expense.

EXAM. 15. Required the dimensions of a cylindrical tankard, to hold one quart of ale measure, that can be made of the least possible quantity of silver, of a given thickness.

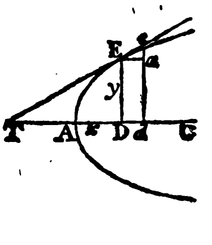
EXAM. 16. To cut the greatest parabola from a given cone.

EXAM. 17. To cut the greatest ellipse from a given cone.

EXAM. 18. To find the value of x when x^x is a mininum.

THE METHOD OF TANGENTS; OR, TO DRAW TANGENTS TO CURVES.

57. THE Method of Tangents, is a method of determining the quantity of the tangent and subtangent of any algebraic curve; the equation of the curve being given. Or, *vice versa*, the nature of the curve, from the tangent given.



If AE be any curve, and E be any point in it, to which it is required to draw a tangent TE. Draw the ordinate ED: then if we can determine the subtangent TD, limited between the ordinate and tangent, in the axis produced, by joining the points T, E, the line TE will be the tangent sought.

58. Let *dae* be another ordinate, indefinitely near to DE, meeting the curve, or tangent produced in *e*; and let E*a* be parallel to the axis AD. Then is the elementary triangle E*ae* similar to the triangle TDE; and

therefore

therefore - $ea : aE :: ED : DT$.
But - - $ea : aE ::$ flux. ED : flux. AD.
Therefore - flux. ED : flux. AD :: DE : DT.
That is, - $\dot{y} : \dot{x} :: y : \frac{y\dot{x}}{\dot{y}} = DT$;

which is therefore the general value of the subtangent sought; where x is the absciss AD, and y the ordinate DE.
Hence we have this general rule.

GENERAL RULE.

59. By means of the given equation of the curve, when put into fluxions, find the value of either $\dot{x}$ or $\dot{y}$, or of $\frac{\dot{x}}{\dot{y}}$ which value substitute for it in the expression $DT = \frac{y\dot{x}}{\dot{y}}$ and, when reduced to its simplest terms, it will be the value of the subtangent sought.

EXAMPLES.

EXAM. 1. Let the proposed curve be that which is defined, or expressed, by the equation $ax^2 + xy^2 - y^3 = 0$.
Here the fluxion of the equation of the curve is $2ax\dot{x} + y^2\dot{x} + 2xy\dot{y} - 3y^2\dot{y} = 0$; then, by transposition, $2ax\dot{x} + y^2\dot{x} = 3y^2\dot{y} - 2xy\dot{y}$; and hence, by division, $\frac{\dot{x}}{\dot{y}} = \frac{3y^2 - 2xy}{2ax + y^2}$; consequently $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - 2xy^2}{2ax + y^2}$. which is the value of the subtangent TD sought.

EXAM. 2. To draw a tangent to a circle; the equation of which is $ax - x^2 = y^2$; where x is the absciss, y the ordinate, and a the diameter.

EXAM. 3. To draw a tangent to a parabola; its equation being $ax = y^2$; where a denotes the parameter of the axis.

EXAM. 4. To draw a tangent to an ellipse; its equation being $c^2 (ax - x^2) = a^2y^2$; where a and c are the two axes.

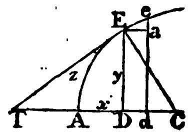
EXAM. 5. To draw a tangent to an hyperbola; its equation being $c^2 (ax + x^2) = a^2y^2$; where a and c are the two axes.

EXAM. 6. To draw a tangent to the hyperbola referred to the asymptote as an axis; its equation being $xy = a^2$; where a^2 denotes the rectangle of the absciss and ordinate answering to the vertex of the curve.

OF

OF RECTIFICATIONS; OR, TO FIND THE LENGTHS OF CURVE LINES.

60. RECTIFICATION, is the finding the length of a curve line, or finding a right line equal to a proposed curve.



By art. 10 it appears, that the elementary triangle Eae, formed by the increments of the absciss, ordinate, and curve, is a right-angled triangle, of which the increment of the curve is the hypothenuse; and therefore the square of the latter is equal to the sum of the squares of the two former; that is, $Ee^2 = Ea^2 + ae^2$. Or, substituting, for the increments, their proportional fluxions, it is $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; where z denotes any curve line AE, x its absciss AD, and y its ordinate DE. Hence this rule.

RULE.

61. From the given equation of the curve put into fluxions, find the value of $\dot{x}^2$ or $\dot{y}^2$, which value substitute instead of it in the equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; then the fluents, being taken, will give the value of z, or the length of the curve, in terms of the absciss or ordinate.

EXAMPLES.

EXAM. 1. To find the length of the arc of a circle, in terms both of the sine, versed sine, tangent, and secant.

The equation of the circle may be expressed in terms of the radius, and either the sine, or the versed sine, or tangent, or secant, &c, of an arc. Let therefore the radius of the circle be CA or CE $= r$, the versed sine AD (of the arc AE) $= x$, the right sine DE $= y$, the tangent TE $= t$, and the secant CT $= s$; then, by the nature of the circle, there arise these equations, viz.

$$y^2 = 2rx - x^2 = \frac{r^2t^2}{r^2 + t^2} = \frac{s^2 - r^2}{s^2} r^2.$$

Then, by means of the fluxions of these equations, with the general fluxional equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, are obtained the following fluxional forms, for the fluxion of the curve; the fluent of any one of which will be the curve itself; viz.

$$\dot{z} = \frac{r\dot{x}}{\sqrt{2rx - xx}} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{r^2\dot{t}}{r^2 + t^2} = \frac{r^2\dot{s}}{\sqrt{s^2 - r^2}}.$$

 Hence

RULE.

63. From the given equation of the curve, find the value either of $\dot{x}$ or of y; which value substitute instead of it in the expression $y\dot{x}$; then the fluent of that expression, being taken, will be the area of the curve sought.

EXAMPLES.

EXAM. 1. To find the area of the common parabola.

The equation of the parabola being $ax = y^2$; where a is the parameter, x the absciss AD, or part of the axis, and y the ordinate DE.

From the equation of the curve is found $y = \sqrt{ax}$. This substituted in the general fluxion of the area $y\dot{x}$ gives $\dot{x}\sqrt{ax}$ or $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$ the fluxion of the parabolic area; and the fluent of this, or $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{ax} = \frac{2}{3}xy$, is the area of the parabola ADE, and which is therefore equal to $\frac{2}{3}$ of its circumscribing rectangle.

EXAM. 2. To square the circle, or find its area.

The equation of the circle being $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$, where a is the diameter; by substitution, the general fluxion of the area $y\dot{x}$, becomes $\dot{x}\sqrt{ax - x^2}$, for the fluxion of the circular area. But as the fluent of this cannot be found in finite terms, the quantity $\sqrt{ax - x^2}$ is thrown into a series, by extracting the root, and then the fluxion of the area becomes

$$\dot{x}\sqrt{ax} \times (1 - \frac{x}{2a} - \frac{x^2}{2.4a^2} - \frac{1.3x^3}{2.4.6a^3} - \frac{1.3.5x^4}{2.4.6.8a^4} - \&c);$$

and then the fluent of every term being taken, it gives

$$x\sqrt{ax} \times (\frac{2}{3} - \frac{1.x}{5a} - \frac{1.x^2}{4.7a^2} - \frac{1.3x^3}{4.6.9a^3} - \frac{1.3.5x^4}{4.6.8.11a^4} - \&c);$$

for the general expression of the semisegment ADE.

And when the point D arrives at the extremity of the diameter, then the space becomes a semicircle, and $x = a$; and then the series above becomes barely

$$a^2(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \frac{1.3.5.}{4.6.8.11} - \&c)$$

for the area of the semicircle whose diameter is a.

EXAM. 3.

EXAM. 3. To find the area of any parabola, whose equation is $a^m z^n = y^{m+n}$.

EXAM. 4. To find the area of an ellipse.

EXAM. 5. To find the area of an hyperbola.

EXAM. 6. To find the area between the curve and asymptote of an hyperbola.

EXAM. 7. To find the like area in any other hyperbola whose general equation is $x^m y^n = a^{m+n}$.

To FIND THE SURFACES OF SOLIDS.

64. IN the solid formed by the rotation of any curve about its axis, the surface may be considered as generated by the circumference of an expanding circle, moving perpendicularly along the axis, but the expanding circumference moving along the arc or curve of the solid. Therefore, as the fluxion of any generated quantity, is produced by drawing the generating quantity into the fluxion of the line or direction in which it moves, the fluxion of the surface will be found by drawing the circumference of the generating circle into the fluxion of the curve. That is, the fluxion of the surface BAE, is equal to $\dot{\text{AE}}$ drawn into the circumference BCEF, whose radius is the ordinate DE.

65. But, if c be $= 3{\cdot}1416$, the circumference of a circle whose diameter is 1, $x =$ AD the absciss, $y =$ DE the ordinate, and $z =$ AE the curve; then $2y =$ the diameter BE, and $2cy =$ the circumference BCEF; also, $\dot{\text{AE}} = \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: therefore $2cy\dot{z}$ or $2cy\sqrt{\dot{x}^2 + \dot{y}^2}$ is the fluxion of the surface. And consequently if, from the given equation of the curve, the value of $\dot{x}$ or $\dot{y}$ be found, and substituted in this expression $2cy\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluent of the expression, being then taken, will be the surface of the solid required.

EXAMPLES.

EXAM. 1. To find the surface of a sphere, or of any segment.

In this case, AE is a circular arc, whose equation is $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$.

The fluxion of this gives $\dot{y} = \frac{a - 2x}{2\sqrt{ax - x^2}}\dot{x} = \frac{a - 2x}{2y}\dot{x}$;

hence $\dot{y}^2 = \frac{a^2 - 4ax + 4x^2}{4y^2}\dot{x}^2 = \frac{a^2 - 4y^2}{4y^2}\dot{x}^2$; consequently $\dot{x}^2 + \dot{y}^2 = \frac{a^2\dot{x}^2}{4y^2}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a\dot{x}}{2y}$.

This value of $\dot{z}$, the fluxion of a circular arc, may be found more easily thus: In the fig. to art. 60, the two triangles EDC, E*ae* are equiangular, being each of them equiangular to the triangle ETC: conseq. ED : EC :: E*a* : E*e*, that is, - - $y : \frac{1}{2}a :: \dot{x} : \dot{z} = \frac{a\dot{x}}{2y}$, the same as before.

The value of $\dot{z}$ being found, by substitution is obtained $2cy\dot{z} = ac\dot{x}$ for the fluxion of the spherical surface, generated by the circular arc in revolving about the diameter AD. And the fluent of this gives acx for the said surface of the spherical segment BAE.

But ac is equal to the whole circumference of the generating circle; and therefore it follows, that the surface of any spherical segment, is equal to the same circumference of the generating circle, drawn into x or AD, the height of the segment.

Also when x or AD becomes equal to the whole diameter a, the expression acx becomes aca or ca^2, or 4 times the area of the generating circle, for the surface of the whole sphere.

And these agree with the rules before found in Mensuration of Solids.

Exam. 2. To find the surface of a spheroid.

Exam. 3. To find the surface of a paraboloid.

Exam. 4. To find the surface of an hyperboloid.

To find the CONTENTS of SOLIDS.

66. Any solid which is formed by the revolution of a curve about its axis (see last fig.), may also be conceived to be generated by the motion of the plane of an expanding circle, moving perpendicularly along the axis. And therefore

fore the area of that circle being drawn into the fluxion of the axis, will produce the fluxion of the solid. That is, $\dot{\text{AD}}$ × area of the circle BCF, whose radius is DE, or diameter BE, is the fluxion of the solid, by art. 9.

67. Hence, if AD $= x$, DE $= y$, $c = 3\cdot1416$; because cy^2 is equal to the area of the circle BCF; therefore $cy^2\dot{x}$ is the fluxion of the solid. Consequently if, from the given equation of the curve, the value of either y^2 or x be found, and that value substituted for it in the expression $cy^2\dot{x}$, the fluent of the resulting quantity, being taken, will be the solidity of the figure proposed.

EXAMPLES.

EXAM. 1. To find the solidity of a sphere, or any segment.

The equation to the generating circle being $y^2 = ax - x^2$, where a denotes the diameter, by substitution, the general fluxion of the solid $cy^2\dot{x}$, becomes $cax\dot{x} - cx^2\dot{x}$, the fluent of which gives $\frac{1}{2}cax^2 - \frac{1}{3}cx^3$, or $\frac{1}{6}cx^2(3a - 2x)$, for the solid content of the spherical segment BAE, whose height AD is x.

When the segment becomes equal to the whole sphere, then $x = a$, and the above expression for the solidity, becomes $\frac{1}{6}ca^3$ for the solid content of the whole sphere.

And these deductions agree with the rules before given and demonstrated in the Mensuration of Solids.

EXAM. 2. To find the solidity of a spheroid.

EXAM. 3. To find the solidity of a paraboloid.

EXAM. 4. To find the solidity of an hyperboloid.

To FIND LOGARITHMS.

68. IT has been proved, art. 23, that the fluxion of the hyperbolic logarithm of a quantity, is equal to the fluxion of the quantity divided by the same quantity. Therefore, when any quantity is proposed, to find its logarithm; take the fluxion of that quantity, and divide it by the same quantity; then take the fluent of the quotient, either in a series or otherwise, and it will be the logarithm sought; when corrected as usual, if need be; that is, the hyperbolic logarithm.

69. But, for any other logarithm, multiply the hyperbolic logarithm, above found, by the modulus of the system, for the logarithm sought.

Note.

Note. The modulus of the hyperbolic logarithms, is 1; and the modulus of the common logarithms, is ·43429448190 &c; and, in general, the modulus of any system, is equal to the logarithm of 10 in that system divided by the number 2·3025850929940&c, which is the hyp. log. of 10. Also, the hyp. log. of any number, is in proportion to the com. log. of the same number, as unity or 1 is to ·43429&c, or as the number 2·302585&c, is to 1; and therefore, if the common log. of any number be multiplied by 2·302585&c, it will give the hyp. log. of the same number; or if the hyp. log. be divided by 2·302585&c, or multiplied by ·43429&c, it will give the common logarithm.

EXAM. 1. To find the log. of $\frac{a+x}{a}$.

Denoting any proposed number z, whose logarithm is required to be found, by the compound expression $\frac{a+x}{a}$, the fluxion of the number $\dot{z}$, is $\frac{\dot{x}}{a}$, and the fluxion of the log. $\frac{\dot{z}}{z} = \frac{\dot{x}}{a+x} = \frac{\dot{x}}{a} - \frac{x\dot{x}}{a^2} + \frac{x^2\dot{x}}{a^3} - \frac{x^3\dot{x}}{a^4} + \&c.$

Then the fluent of these terms give the logarithm of z or logarithm of $\frac{a+x}{a} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

Writing $-x$ for x, gives log. $\frac{a-x}{a} = -\frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

Div. these numb. and subtr. their logs. gives $\Big\}$ log. $\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5}$ &c.

Also, because $\frac{a}{a\pm x} = 1 \div \frac{a\pm x}{a}$, or log. $\frac{a}{a\pm x} = 0 - \text{log.} \frac{a\pm x}{a}$;

therefore log. of $\frac{a}{a+x}$ is $-\frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c,

and the log. of $\frac{a}{a-x}$ is $+\frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c,

the prod. gives log. $\frac{a^2}{a^2-x^2} = \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{3a^6} + \&c.$

Now, for an example in numbers, suppose it were required to compute the common logarithm of the number 2. This will be best done by the series,

$$\text{log. of } \frac{a+x}{a-x} = 2m \times \left(\frac{x}{a} + \frac{x^3}{3a^3} + \frac{x^5}{5a^5} + \frac{x^7}{7a^7}\ \&c.\right.$$

Making

Making $\frac{a+x}{a-x}=2$, gives $a=3x$; conseq. $\frac{x}{a}=\frac{1}{3}$, and $\frac{x^2}{a^2}$ $=\frac{1}{9}$, which is the constant factor for every succeeding term; also, $2m=2\times{}$·43429448190 = ·868588964; therefore the calculation will be conveniently made, by first dividing this number by 3, then the quotients successively by 9, and lastly these quotients in order by the respective numbers 1, 3, 5, 7, 9, &c, and after that, adding all the terms together, as follows:

3)	·868588964				
9)	289529654	1)	·289529654	(	·289529654
9)	32169962	3)	32169962	(	10723321
9)	3574440	5)	3574440	(	714888
9)	397160	7)	397160	(	56737
9)	44129	9)	44129	(	4903
9)	4903	11)	4903	(	446
9)	545	13)	545	(	42
9)	61	15)	61	(	4

Sum of the terms gives log. 2 = ·301029995

EXAM. 2. To find the log. of $\frac{a+x}{b}$.

EXAM. 3. To find the log. of $a-x$.

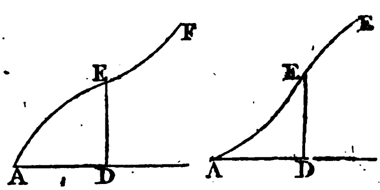
EXAM. 4. To find the log. of 3.

EXAM. 5. To find the log. of 5.

EXAM. 6. To find the log. of 11.

TO FIND THE POINTS OF INFLEXION, OR OF CONTRARY FLEXURE IN CURVES.

70. THE Point of Inflexion in a curve, is that point of it which separates the concave from the convex part, lying between the two; or where the curve changes from concave to convex, or from convex to concave, on the same side of the curve. Such as the point E in the annexed figures, where the former of the two is concave



towards

towards the axis AD, from A to E, but convex from E to F; and on the contrary, the latter figure is convex from A to E, and concave from E to F.

71. From the nature of curvature, as has been remarked before at art. 28, it is evident, that when a curve is concave towards an axis, then the fluxion of the ordinate decreases, or is in a decreasing ratio, with regard to the fluxion of the absciss; but, on the contrary, that it increases, or is in an increasing ratio to the fluxion of the absciss, when the curve is convex towards the axis; and consequently those two fluxions are in a constant ratio at the point of inflexion, where the curve is neither convex nor concave; that is, $\dot{x}$ is to $\dot{y}$ in a constant ratio, or $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ is a constant quantity. But constant quantities have no fluxion, or their fluxion is equal to nothing; so that in this case, the fluxion of $\frac{\dot{y}}{\dot{x}}$ or of $\frac{\dot{x}}{\dot{y}}$ is equal to nothing. And hence we have this general rule:

72. Put the given equation of the curve into fluxions; from which find either $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$. Then take the fluxion of this ratio, or fraction, and put it equal to 0 or nothing; and from this last equation find also the value of the same $\frac{\dot{x}}{\dot{y}}$ or $\frac{\dot{y}}{\dot{x}}$. Then put this latter value equal to the former, which will form an equation; from which, and the first given equation of the curve, x and y will be determined, being the absciss and ordinate answering to the point of inflexion in the curve, as required.

EXAMPLES.

EXAM. 1. To find the point of inflexion in the curve whose equation is $ax^2 = a^2y + x^2y$.

This equation in fluxions is $2ax\dot{x} = a^2\dot{y} + 2xy\dot{x} + x^2\dot{y}$, which gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2+x^2}{2ax-2xy}$. Then the fluxion of this quantity made $= 0$, gives $2x\dot{x}(ax-xy) = (a^2+x^2) \times (a\dot{x}-\dot{x}y-x\dot{y})$; and this again gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2+x^2}{a^2-x^2} \times \frac{x}{a-y}$.

Lastly, this value of $\frac{\dot{x}}{\dot{y}}$ being put equal the former, gives

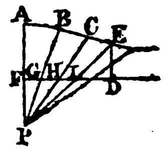
$\frac{a^2+x^2}{a^2-x^2}$

$\frac{a^2+x^2}{a^2-x^2}\cdot\frac{x}{a-y}=\frac{a^2+x^2}{2x}\cdot\frac{1}{a-y}$; and hence $2x^2=a^2-x^2$, or $3x^2=a^2$, and $x=a\sqrt{\frac{1}{3}}$, the absciss.

Hence also, from the original equation, - - - - - -
$y=\frac{ax^2}{a^2+x^2}=\frac{\frac{1}{3}a^3}{\frac{4}{3}a^2}=\frac{1}{4}a$, the ordinate of the point of inflexion sought.

EXAM. 2. To find the point of inflexion in a curve defined by the equation $ay=a\sqrt{ax^2+xx}$.

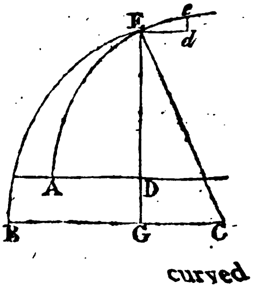
EXAM. 3. To find the point of inflexion in a curve defined by the equation $ay^2=a^2x+x^3$.



EXAM. 4. To find the point of inflexion in the Conchoid of Nicomedes, which is generated or constructed in this manner: From a fixed point P, which is called the pole of the conchoid, draw any number of right lines PA, PB, PC, PE, &c, cutting the given line FD in the points F, G, H, I, &c: then make the distances FA, GB, HC, IE, &c, equal to each other, and equal to a given line; then the curve line ABCE &c, will be the conchoid; a curve so called by its inventor Nicomedes.

TO FIND THE RADIUS OF CURVATURE OF CURVES.

73. THE Curvature of a Circle is constant, or the same in every point of it, and its radius is the radius of curvature. But the case is different in other curves, every one of which has its curvature continually varying, either increasing or decreasing, and every point having a degree of curvature peculiar to itself; and the radius of a circle which has the same curvature with the curve at any given point, is the radius of curvature at that point; which radius it is the business of this chapter to find.



74. Let AEe be any curve, concave towards its axis AD; draw an ordinate DE to the point E, where the curvature is to be found; and suppose EC perpendicular to the curve, and equal to the radius of curvature sought, or equal to the radius of a circle having the same curvature there, and with that radius describe the said equally-

curved

curved circle BEe; lastly, draw Ed parallel to AD, and de parallel and indefinitely near to DE: thereby making Ed the fluxion or increment of the absciss AD, also de the fluxion of the ordinate DE, and Ee that of the curve AE. Then put $x =$ AD, $y =$ DE, $z =$ AE, and $r =$ CE the radius of curvature; then is E$d = \dot{x}$, $de = \dot{y}$, and E$e = \dot{z}$.

Now, by sim. triangles, the three lines Ed, de, Ee,
or $\dot{x}$, $\dot{y}$, $\dot{z}$,
are respectively as the three - - - GE, GC, CE;
therefore - - - - - - - - - $\text{GC} \cdot \dot{x} = \text{GE} \cdot \dot{y}$;
and the flux. of this eq. is $\text{GC} \cdot \ddot{x} + \dot{\text{GC}} \cdot \dot{x} = \text{GE} \cdot \ddot{y} + \dot{\text{GE}} \cdot \dot{y}$,
or, because $\dot{\text{GC}} = -\dot{\text{BG}}$, it is $\text{GC} \cdot \ddot{x} - \dot{\text{BG}} \cdot \dot{x} = \text{GE} \cdot \ddot{y} + \dot{\text{GE}} \cdot \dot{y}$.

But since the two curves AE and BE have the same curvature at the point E, their abscisses and ordinates have the same fluxions at that point, that is, Ed or $\dot{x}$ is the fluxion both of AD and BG, and de or $\dot{y}$ is the fluxion both of DE and GE. In the equation above therefore substitute $\dot{x}$ for $\dot{\text{BG}}$, and $\dot{y}$ for $\dot{\text{GE}}$, and it becomes

$$\text{GC}\ddot{x} - \dot{x}\dot{x} = \text{GE}\ddot{y} + \dot{y}\dot{y},$$
$$\text{or } \text{GC}\ddot{x} - \text{GE}\ddot{y} = \dot{x}^2 + \dot{y}^2 = \dot{z}^2.$$

Now multiply the three terms of this equation respectively by these three quantities, $\frac{\dot{y}}{\text{GC}}$, $\frac{\dot{x}}{\text{GE}}$, $\frac{\dot{z}}{\text{CE}}$, which are all equal,

and it becomes - - - $\dot{y}\ddot{x} - \dot{x}\ddot{y} = \frac{\dot{z}^3}{\text{CE}}$, or $\frac{\dot{z}^3}{r}$;

and hence is found $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, for the general value of the radius of curvature, for all curves whatever, in terms of the fluxions of the absciss and ordinate.

75. Further, as in any case either x or y may be supposed to flow equably, that is, either $\dot{x}$ or $\dot{y}$ constant quantities, or $\ddot{x}$ or $\ddot{y}$ equal to nothing, it follows that, by this supposition, either of the terms in the denominator, of the value of r, may be made to vanish. Thus, when $\dot{x}$ is supposed constant, $\ddot{x}$ being then $= 0$, the value of r is barely - - - - - $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or r is $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is constant.

EXAMPLES.

EXAM. 1. To find the radius of curvature to any point of

of a parabola, whose equation is $ax = y^2$, its vertex being A, and axis AD.

Now, the equation to the curve being $ax = y^2$, the fluxion of it is $a\dot{x} = 2y\dot{y}$; and the fluxion of this again is $a\ddot{x} = 2\dot{y}^2$, supposing $\dot{y}$ constant; hence then r or

$$\frac{\dot{z}^3}{\dot{y}\ddot{x}} \text{ or } \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}\ddot{x}} \text{ is } = \frac{(a^2+4y^2)^{\frac{3}{2}}}{2a^2} \text{ or } \frac{(a+4x)^{\frac{3}{2}}}{2\sqrt{a}},$$

for the general value of the radius of curvature at any point E, the ordinate to which cuts off the absciss AD $= x$.

Hence, when the absciss x is nothing, the last expression becomes barely $\frac{1}{2}a = r$, for the radius of curvature at the vertex of the parabola; that is, the diameter of the circle of curvature at the vertex of a parabola, is equal to a, the parameter of the axis.

EXAM. 2. To find the radius of curvature of an ellipse, whose equation is $a^2y^2 = c^2 . \overline{ax - x^2}$.

$$\text{Ans. } r = \frac{(a^2c^2 + 4(a^2 - c^2) \times (ax - x^2))^{\frac{3}{2}}}{2a^4c}.$$

EXAM. 3. To find the radius of curvature of an hyperbola, whose equation is $a^2y^2 = c^2 . \overline{ax + x^2}$.

EXAM. 4. To find the radius of curvature of the cycloid.

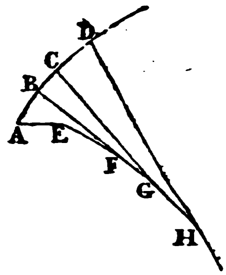
Ans. $r = 2\sqrt{aa - ax}$, where x is the absciss, and a the diameter of the generating circle.

OF INVOLUTE AND EVOLUTE CURVES.

76. AN Evolute is any curve supposed to be evolved or opened, which having a thread wrapped close about it, fastened at one end, and beginning to evolve or unwind the thread from the other end, keeping always tight stretched the part which is evolved or wound off: then this end of the thread will describe another curve, called the Involute. Or, the same involute is described in the contrary way, by wrapping the thread about the curve of the evolute, keeping it at the same time always stretched.

77. Thus

77. Thus, if EFGH be any curve, and AE be either a part of the curve, or a right line: then if a thread be fixed to the curve at H, and be wound or plied close to the curve, &c, from H to A, keeping the thread always stretched tight; the other end of the thread will describe a certain curve ABCD, called an Involute; the first curve EFGH being its evolute. Or, if the thread, fixed at H, be unwound from the curve, beginning at A, and keeping it always tight, it will describe the same involute ABCD.

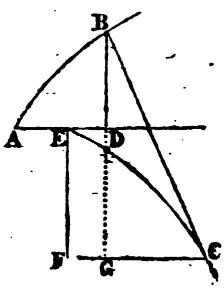


78. If AE, DF, CG, DH, &c, be any positions of the thread, in evolving or unwinding; it follows, that these parts of the thread are always the radii of curvature, at the corresponding points, A, B, C, D; and also equal to the corresponding lengths AE, AEF, AEFG, AEFGH, of the evolute; that is,

AE $=$ AE is the radius of curvature to the point A,
BF $=$ AF is the radius of curvature to the point B,
CG $=$ AG is the radius of curvature to the point C,
DH $=$ AH is the radius of curvature to the point D.

79. It also follows, from the premises, that any radius of curvature, BF, is perpendicular to the involute at the point B, and is a tangent to the evolute curve at the point F. Also, that the evolute is the locus of the centre of curvature of the involute curve.

80. Hence, and from art. 74, it will be easy to find one of these curves, when the other is given. To this purpose, put



$x =$ AD, the absciss of the involute,
$y =$ DB, an ordinate to the same,
$z =$ AB, the involute curve,
$r =$ BC, the radius of curvature,
$v =$ EF, the absciss of the evolute EC,
$=$ FC, the ordinate of the same, and
$=$ AE, a certain given line.

Then,

Then, by the nature of the radius of curvature, it is

$$r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \text{BC} = \text{AE} + \text{EC}; \text{ also, by sim. triangles,}$$

$$\dot{z} : \dot{x} :: r : \text{GB} = \frac{r\dot{x}}{\dot{z}} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}};$$

$$\dot{z} : \dot{y} :: r : \text{GC} = \frac{r\dot{y}}{\dot{z}} = \frac{\dot{y}\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}.$$

Hence $\text{EF} = \text{GB} - \text{DB} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} - y = v;$

and $\text{FC} = \text{AD} - \text{AE} + \text{GC} = x - a + \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = u;$

which are the values of the absciss and ordinate of the evolute curve EC; from which therefore these may be found, when the involute is given.

On the contrary, if v and u, or the evolute, be given: then, putting the given curve $\text{EC} = s$, since $\text{CB} = \text{AE} + \text{EC}$, or $r = a + s$, this gives r the radius of curvature. Also, by similar triangles, there arise these proportions, viz.

$$\dot{s} : \dot{v} :: r : \frac{r\dot{v}}{\dot{s}} = \frac{a+s}{\dot{s}}\dot{v} = \text{GB},$$

and $\dot{s} : \dot{u} :: r : \frac{r\dot{u}}{\dot{s}} = \frac{a+s}{\dot{s}}\dot{u} = \text{GC};$

theref. $\text{AD} = \text{AE} + \text{FC} - \text{GC} = a + u - \frac{a+s}{\dot{s}}\dot{u} = x,$

and $\text{DB} = \text{GB} - \text{GD} = \text{GB} - \text{EF} = \frac{a+s}{\dot{s}}\dot{v} - v = y;$

which are the absciss and ordinate of the involute curve, and which may therefore be found, when the evolute is given. Where it may be noted, that $\dot{s}^2 = \dot{v}^2 + \dot{u}^2$, and $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$. Also, either of the quantities x, y, may be supposed to flow equably, in which case the respective second fluxion, $\ddot{x}$ or $\ddot{y}$, will be nothing, and the corresponding term in the denominator $\dot{y}\ddot{x} - \dot{x}\ddot{y}$ will vanish, leaving only the other term in it; which will have the effect of rendering the whole operation simpler.

81. EXAMPLES.

EXAM. 1. To determine the nature of the curve by whose evolution the common parabola AB is described.

Here

Here the equation of the given involute AB, is $cx = y^2$ where c is the parameter of the axis AD. Hence then $\dot{y} = \sqrt{cx}$, and $\dot{y} = \frac{1}{2}\dot{x}\sqrt{\frac{c}{x}}$, also $\ddot{y} = \frac{-\dot{x}^2}{4x}\sqrt{\frac{c}{x}}$ by making $\dot{x}$ constant. Consequently the general values of v and u, or of the absciss and ordinate, EF and FC, above given, become, in that case,

$$\text{EF} = v = \frac{\dot{z}^2}{-\ddot{y}} - y = \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}} - y = 4x\sqrt{\frac{x}{c}}; \text{ and}$$

$$\text{FC} = u = x - a + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}} = 3x + \tfrac{1}{2}c - a.$$

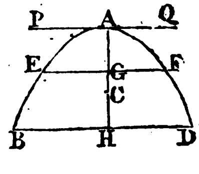
But the value of the quantity a or AE, by exam. 1 to art. 75, was found to be $\frac{1}{2}c$; consequently the last quantity, FC or u, is barely $= 3x$.

Hence then, comparing the values of v and u, there is found $3v\sqrt{c} = 4u\sqrt{x}$, or $27cv^2 = 16u^3$; which is the equation between the absciss and ordinate of the evolute curve EC, showing it to be the semicubical parabola.

EXAM. 2. To determine the evolute of the common cycloid. Ans. another cycloid, equal to the former.

To Find the CENTRE of GRAVITY.

82. By referring to prop. 42, &c, in Mechanics, it is seen what are the principles and nature of the Centre of Gravity in any figure, and how it is generally expressed. It there appears, that if PAQ be a line, or plane, drawn through any point, as suppose the vertex of any body, or figure, ABD, and if s denote any section EF of the figure, $d = $ AG, its distance below PQ, and $b =$ the whole body or figure ABD; then the distance AC, of the centre of gravity below PQ, is universally denoted by $\frac{\text{sum of all the } ds}{b}$; whether ABD be a line, or a plane surface, or a curve superficies, or a solid.



But

But the sum of all the ds, is the same as the fluent of $d\dot{b}$, and b is the same as the fluent of $\dot{b}$; therefore the general expression for the distance of the centre of gravity, is AC $= \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent } x\dot{b}}{b}$; putting $x = d$ the variable distance AG. Which will divide into the following four cases.

83. CASE 1. When AE is some line, as a curve suppose. In this case $\dot{b}$ is $= \dot{z}$ or $\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluxion of the curve; and $b = z$: theref. AC $= \frac{\text{fluent of } x\dot{z}}{z} = \frac{\text{fluent of } x\sqrt{\dot{x}^2+\dot{y}^2}}{z}$ is the distance of the centre of gravity in a curve.

84. CASE 2. When the figure ABD is a plane; then $\dot{b} = y\dot{x}$; therefore the general expression becomes AC $= \frac{\text{fluent of } yx\dot{x}}{\text{fluent of } y\dot{x}}$ for the distance of the centre of gravity in a plane.

85. CASE 3. When the figure is the superficies of a body generated by the rotation of a line AEB, about the axis AH. Then, putting $c = 3{\cdot}14159$ &c, $2cy$ will denote the circumference of the generating circle, and $2cy\dot{z}$ the fluxion of the surface; therefore AC $= \frac{\text{fluent of } 2cyx\dot{z}}{\text{fluent of } 2cy\dot{z}} = \frac{\text{fluent of } yx\dot{z}}{\text{fluent of } y\dot{z}}$ will be the distance of the centre of gravity for a surface generated by the rotation of a curve line z.

86. CASE 4. When the figure is a solid generated by the rotation of a plane ABH, about the axis AH.

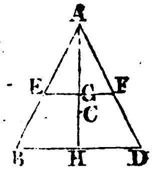
Then, putting $c = 3{\cdot}14159$ &c, it is cy^2 = the area of the circle whose radius is y, and $cy^2\dot{x} = \dot{b}$, the fluxion of the solid; therefore

$$\text{AC} = \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent of } cy^2x\dot{x}}{\text{fluent of } cy^2\dot{x}} = \frac{\text{fluent of } y^2x\dot{x}}{\text{fluent of } y^2\dot{x}}$$ is the distance of the centre of gravity below the vertex in a solid.

87. EXAMPLES.

EXAM. 1. Let the figure proposed be the isosceles triangle ABD.

It is evident that the centre of gravity C, will be some-



where in the perpendicular AH. Now, if a denote AH, c = BD, x = AG, and y = EF any line parallel to the base BD: then as $a : c :: x : y = \frac{cx}{a}$; therefore, by the 2d Case, AC $= \frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \frac{\text{fluent } x^2\dot{x}}{\text{fluent } x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2} = \frac{2}{3}x = \frac{2}{3}$AH, when x becomes = AH: consequently CH = $\frac{1}{3}$AH.

In like manner, the centre of gravity of any other plane triangle, will be found to be at $\frac{1}{3}$ of the altitude of the triangle; the same as it was found in prop. 43, Mechanics.

EXAM 2. In a parabola; the distance from the vertex is $\frac{3}{5}x$, or $\frac{3}{5}$ of the axis.

EXAM. 3. In a circular arc; the distance from the centre of the circle, is $\frac{cr}{a}$; where a denotes the arc, c its chord, and r the radius.

EXAM. 4. In a circular sector; the distance from the centre of the circle, is $\frac{2cr}{3a}$: where a, c, r, are the same as in exam. 3.

EXAM 5. In a circular segment; the distance from the centre of the circle is $\frac{c^3}{12a}$; where c is the chord, and a the area, of the segment.

EXAM. 6. In a cone, or any other pyramid; the distance from the vertex is $\frac{3}{4}x$, or $\frac{3}{4}$ of the altitude.

EXAM. 7. In the semisphere, or semispheroid; the distance from the centre is $\frac{3}{8}r$, or $\frac{3}{8}$ of the radius: and the distance from the vertex $\frac{5}{8}$ of the radius.

EXAM. 8. In the parabolic conoid; the distance from the base is $\frac{1}{3}x$, or $\frac{1}{3}$ of the axis. And the distance from the vertex $\frac{2}{3}$ of the axis.

EXAM. 9. In the segment of a sphere, or of a spheroid; the distance from the base is $\frac{2a - x}{6a - 4x}x$; where x is the height of the segment, and a the whole axis, or diameter of the sphere.

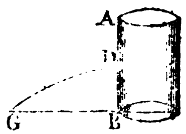
EXAM. 10. In the hyperbolic conoid; the distance from the base is $\frac{2a + x}{6a + 4x}x$; where x is the height of the conoid, and a the whole axis or diameter.

PRACTICAL QUESTIONS.

QUESTION I.

A LARGE vessel, of 10 feet, or any other given depth, and of any shape, being kept constantly full of water, by means of a supplying cock, at the top; it is proposed to assign the place where a small hole must be made in the side of it, so that the water may spout through it to the greatest distance on the plane of the base.

Let AB denote the height or side of the vessel; D the required hole in the side, from which the water spouts, in the parabolic curve DG, to the greatest distance BG, on the horizontal plane.

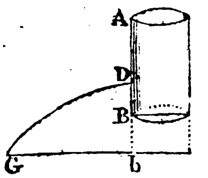


By the scholium to prop. 68, Hydraulics, the distance BG is always equal to $2\sqrt{AD \cdot DB}$, which is equal to $2\sqrt{x(a-x)}$ or $2\sqrt{ax-x^2}$, if a be put to denote the whole height AB of the vessel, and $x =$ AD the depth of the hole. Hence $2\sqrt{ax-x^2}$, or $ax - x^2$, must be a maximum. In fluxions, $a\dot{x} - 2x\dot{x} = 0$, or $a - 2x = 0$, and $2x = a$, or $x = \frac{1}{2}a$. So that the hole D must be in the middle between the top and bottom; the same as before found at the end of the scholium above quoted.

QUESTION II.

If the same vessel, as in Quest. 1, stand on high, with its bottom a given height above a horizontal plane below; it is proposed to determine where the small hole must be made, so as to spout farthest on the said plane.

Let the annexed figure represent the vessel as before, and bG the greatest distance spouted by the fluid, DG, on the plane bG.



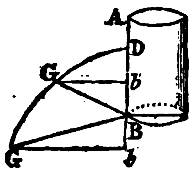
Here, as before, $bG = 2\sqrt{AD \cdot Db} = 2\sqrt{x(c-x)} = 2\sqrt{cx-x^2}$, by putting $Ab = c$, and $AD = x$. So that $2\sqrt{cx-x^2}$ or $cx - x^2$ must be a maximum. And hence, like as in the former question, $x = \frac{1}{2}c = \frac{1}{2}Ab$. So that the hole D must be made in the

 middle

middle between the top of the vessel, and the given plane, that the water may spout farthest.

QUESTION III.

But if the same vessel, as before, stand on the top of an inclined plane, making a given angle, as suppose of 30 degrees, with the horizon; it is proposed to determine the place of the small hole, so as the water may spout the farthest on the said inclined plane.



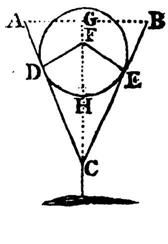
Here again (D being the place of the hole, and BG the given inclined plane), $bG = 2\sqrt{AD \cdot Db} = 2\sqrt{x(a - x \pm z)}$, putting $z = Bb$, and, as before, $a = AB$, and $x = AD$. Then bG must still be a maximum, as also Bb, being in a given ratio to the maximum BG, on account of the given angle B. Therefore $ax - x^2 \pm az$, as well as z, is a maximum. Hence, by art. 54 of the Fluxions, $a\dot{x} - 2x\dot{x} \pm z\dot{x} = 0$, or $a - 2x \pm z = 0$; conseq. $\pm z = 2x - a$; and hence $bG = 2\sqrt{x(a - x \pm z)}$ becomes barely $2x$. But as the given angle GBb is $= 30°$, the sine of which is $\frac{1}{2}$; therefore $BG = 2Bb$ or $2z$, and $bG^2 = BG^2 - Bb^2 = 3z^2 = 3(2x - a)^2$, or $bG = \pm(2x - a)\sqrt{3}$.

Putting now these two values of bG equal to each other, gives the equation $2x = \pm(2x - a)\sqrt{3}$, from which is found $x = \dfrac{\frac{1}{2}a\sqrt{3}}{\sqrt{3} \pm 1} = \dfrac{3 \pm \sqrt{3}}{4}a$, the value of AD required.

Note. In the Select Exercises, page 269, this answer is brought out $\dfrac{6 + \sqrt{6}}{10}a$, by taking the velocity proportional to the root of *half* the altitude only.

QUESTION IV.

It is required to determine the size of a ball, which, being let fall into a conical glass full of water, shall expel the most water possible from the glass; its depth being 6, and diameter 5 inches.



Let ABC represent the cone of the glass, and DHE the ball, touching the sides in the points D and E, the centre of the ball being at some points F in the axis GC of the cone.

Put

Put AG = GB = $2\frac{1}{2} = a$,
CC = $6 = b$,
AG = $\sqrt{AG^2 + GC^2} = 6\frac{1}{2} = c$,
AD = FE = FH = x the radius of the ball.

The two triangles ACG and DCF are equiangular; theref. AG : AC :: DF : FC, that is, $a : c :: x : \frac{cx}{a}$ = FC; hence GF = GC − FC = $b - \frac{cx}{a}$, and GH = GF + FH = $b + x - \frac{cx}{a}$, the height of the segment immersed in the water. Then (by rule 1 for the spherical segment, page 51), the content of the said immersed segment will be (6DF − 2GH) × GH² × ·5236 = $(2x - b + \frac{cx}{a}) \times (x + b - \frac{cx}{a})^2 \times 1{\cdot}0472$, which must be a maximum by the question; the fluxion of this made = 0, and divided by $2\dot{x}$ and the common factors, gives $\frac{2a + c}{a} \times (b - \frac{c - a}{a}x) - (\frac{2a + c}{a}x - b) \times \frac{c - a}{a} \times 2 = 0$; this reduced gives $x = \frac{abc}{(c - a) \times (c + 2a)} = 2\frac{1}{92}$, the radius of the ball. Consequently its diameter is $4\frac{1}{46}$ inches, as required.

PRACTICAL EXERCISES CONCERNING FORCES; WITH THE RELATION BETWEEN THEM AND THE TIME, VELOCITY, AND SPACE DESCRIBED.

Before entering on the following problems, it will be convenient here, to lay down a synopsis of the theorems which express the several relations between any forces, and their corresponding times, velocities, and spaces, described; which are all comprehended in the following 12 theorems, as collected from the principles in the foregoing parts of this work.

Let f, F, be any two constant accelerative forces, acting on any body, during the respective times t, T, at the end of which are generated the velocities v, V, and described the spaces s, S. Then, because the spaces are as the times and velocities conjointly, and the velocities as the forces and times; we shall have,

I. In

I. *In Constant Forces.*

$$1.\quad \frac{s}{S} = \frac{tv}{TV} = \frac{t^2f}{T^2F} = \frac{v^2F}{V^2f}.$$

$$2.\quad \frac{v}{V} = \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fs}{FS}}.$$

$$3.\quad \frac{t}{T} = \frac{Fv}{fV} = \frac{sV}{Sv} = \sqrt{\frac{Fs}{fS}}.$$

$$4.\quad \frac{f}{F} = \frac{Tv}{tV} = \frac{T^2s}{t^2S} = \frac{v^2S}{V^2s}.$$

And if one of the forces, as F, be the force of gravity at the surface of the earth, and be called 1, and its time T be $= 1''$; then it is known by experiment that the corresponding space S is $= 16\frac{1}{12}$ feet, and consequently its velocity $V = 2S = 32\frac{1}{6}$, which call $2g$, namely, $g = 16\frac{1}{12}$ feet, or 193 inches. Then the above four theorems, in this case, become as here below:

$$5.\quad s = \tfrac{1}{2}tv = gft^2 = \frac{v^2}{4gf}.$$

$$6.\quad v = \frac{2s}{t} = 2gft = \sqrt{4gfs}.$$

$$7.\quad t = \frac{2s}{v} = \frac{v}{2gf} = \sqrt{\frac{s}{gf}}.$$

$$8.\quad f = \frac{v}{2gt} = \frac{s}{gt^2} = \frac{v^2}{4gs}.$$

And from these are deduced the following four theorems, for variable forces, viz.

II. *In Variable Forces.*

$$9.\quad \dot{s} = v\dot{t} = \frac{v\dot{v}}{2gf}.$$

$$10.\quad \dot{v} = 2gf\dot{t} = \frac{2gf\dot{s}}{v}.$$

$$11.\quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{2gf}.$$

$$12.\quad f = \frac{v\dot{v}}{2g\dot{s}} = \frac{\dot{v}}{2g\dot{t}}.$$

In

In these last four theorems, the force f, though variable, is supposed to be constant for the indefinitely small time $\dot{t}$, and they are to be used in all cases of variable forces, as the former ones in constant forces; namely from the circumstances of the problem under consideration, an expression is deduced for the value of the force f, which being substituted in one of these theorems, that may be proper to the case in hand; the equation thence resulting will determine the corresponding values of the other quantities, required in the problem.

When a motive force happens to be concerned in the question, it may be proper to observe, that the motive force m, of a body, is equal to fq, the product of the accelerative force, and the quantity of matter in it q; and the relation between these three quantities being universally expressed by this equation $m = qf$, it follows that, by means of it, any one of the three may be expelled out of the calculation, or else brought into it.

Also, the momentum, or quantity of motion in a moving body, is qv, the product of the velocity and matter.

It is also to be observed, that the theorems equally hold good for the destruction of motion and velocity, by means of retarding forces, as for the generation of the same, by means of accelerating forces.

And to the following problems, which are all resolved by the application of these theorems, it has been thought proper to subjoin their solutions, for the better information and convenience of the student.

PROBLEM I.

To determine the time and velocity of a body descending, by the force of gravity, down an inclined plane; the length of the plane being 20 feet, and its height 1 foot.

Here, by Mechanics, the force of gravity being to the force down the plane, as the length of the plane is to its height, therefore as $20 : 1 :: 1$ (the force of gravity) $: \frac{1}{20} = f$, the force on the plane.

Therefore, by theor. 6, v or $\sqrt{4gfs}$ is $\sqrt{4 \times 16\frac{1}{12} \times \frac{1}{20} \times 20} = \sqrt{4 \times 16\frac{1}{12}} = 2 \times 4\frac{1}{96}$ or $8\frac{1}{48}$ feet nearly, the last velocity per second. And,

By theor. 7, t or $\sqrt{\frac{s}{gf}}$ is $\sqrt{\frac{20}{16\frac{1}{12} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{12}}} = \frac{20}{4\frac{1}{96}}$ $= 4\frac{76}{77}$ seconds, the time of descending.

PROBLEM II.

If a cannon ball be fired with a velocity of 1000 feet per second, up a smooth inclined plane, which rises 1 foot in 20: it is proposed to assign the length which it will ascend up the plane, before it stops and begins to return down again, and the time of its ascent.

Here $f = \frac{1}{20}$ as before.

Then, by theor. 5, $s = \frac{v^2}{4gf} = \frac{1000^2}{4 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{60000000}{193}$ $= 310880\frac{160}{193}$ feet, or nearly 59 miles, the distance moved.

And, by theor. 7, $t = \frac{v}{2gf} = \frac{1000}{2 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{120000}{193} =$ $621''\frac{147}{193} = 10'\ 21''\frac{147}{193}$, the time of ascent.

PROBLEM III.

If a ball be projected up a smooth inclined plane, which rises 1 foot in 10, and ascend 100 feet before it stop: required the time of ascent, and the velocity of projection.

First, by theor. 6, $v = \sqrt{4gfs} = \sqrt{4 \times 16\frac{1}{12} \times \frac{1}{10} \times 100} = 8\frac{1}{48}\sqrt{10} = 25{\cdot}36408$ feet per second, the velocity.

And, by theor. 7, $t = \sqrt{\frac{s}{gf}} = \sqrt{\frac{100}{16\frac{1}{12} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{96}}\sqrt{10} =$ $\frac{192}{77}\sqrt{10} = 7{\cdot}88516$ seconds, the time in motion.

PROBLEM IV.

If a ball be observed to ascend up a smooth inclined plane, 100 feet in 10 seconds, before it stop, to return back again: required the velocity of projection, and the angle of the plane's inclination.

First, by theor. 6, $v = \frac{2s}{t} = \frac{200}{10} = 20$ feet per second, the velocity.

And, by theor. 8, $f = \frac{s}{gt^2} = \frac{100}{16\frac{1}{12} \times 100} = \frac{12}{193}$. That is, the length of the plane is to its height, as 193 to 12.

Therefore 193 : 12 : : 100 : 6·2176 the height of the plane, or the sine of elevation to radius 100, which answers to 3° 34′, the angle of elevation of the plane.

PROBLEM

PROBLEM V.

By a mean of several experiments, I have found, that a cast iron ball, of 2 inches diameter, fired perpendicularly into the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 feet per second, penetrated 13 inches deep into its substance. It is proposed then to determine the time of the penetration, and the resisting force of the wood, as compared to the force of gravity, supposing that force to be a constant quantity.

First, by theor. 7, $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$ part of a second, the time in penetrating.

And, by theor. 8, $f = \frac{v^2}{4gs} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{81000000}{13 \times 193}$ $= 32284$. That is, the resisting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its density or specific gravity. For, since f is as $\frac{v^2}{s}$ by theor. 4, the density and size of the ball remaining the same; if the density, or specific gravity, n, vary, and all the rest be constant, it is evident that f will be as n; and therefore f as $\frac{nv^2}{s}$ when the size of the ball only is constant. But when only the diameter d varies, all the rest being constant, the force of the blow will vary as d^3, or as the magnitude of the ball; and the resisting surface, or force of resistance, varies as d^2; therefore f is as $\frac{d^3}{d^2}$, or as d only when all the rest are constant. Consequently f is as $\frac{dnv^2}{s}$ when they are all variable.

And so $\frac{f}{F} = \frac{dnv^2S}{DNV^2s}$, and $\frac{s}{S} = \frac{dnv^2F}{DNV^2f}$; where f denotes the strength or firmness of the substance penetrated, and is here supposed to be the same, for all balls and velocities, in the same substance, which is either accurately or nearly so. See page 264, &c, of my Tracts.

Hence, taking the numbers in the problem, it is - - -
$f = \frac{dnv^2}{s} = \frac{\frac{2}{12} \times 7\frac{1}{3} \times 1500^2}{\frac{13}{12}} = \frac{44 \times 1500^2}{39} = 2538462$
the value of f for elm wood. Where the specific gravity of the

the ball is taken $7\frac{1}{3}$, which is a little less than that of solid cast iron, as it ought, on account of the air bubble which is found in all cast balls.

PROBLEM VI.

To find how far a 24lb ball of cast iron will penetrate into a block of sound elm, when fired with a velocity of 1600 feet per second.

HERE, because the substance is the same as in the last problem, both of the balls and wood, $N = n$, and $F = f$; therefore $\frac{S}{s} = \frac{DV^2}{dv^2}$, or $S = \frac{DV^2 s}{dv^2} = \frac{5\cdot55 \times 1600^2 \times 13}{2 \times 1500^2} =$ $41\frac{2}{45}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins (vol. i. p. 273, *of his works*), *that an* 18-*pounder ball, fired with a velocity of* 1200 *feet per second, penetrated* 34 *inches into sound dry oak. It is required then to ascertain the comparative strength or firmness of oak and elm.*

THE diameter of an 18lb ball is 5·04 inches = D. Then, by the numbers given in this problem for oak, and in prob. 5, for elm, we have - - - - - - - - - - - - - - - -
$\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 34}{5\cdot04 \times 1200^2 \times 13} = \frac{100 \times 17}{5\cdot04 \times 16 \times 13} = \frac{1700}{1048}$
or $= \frac{8}{5}$ nearly.

From which it would seem, that elm timber resists more than oak, in the ratio of about 8 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be suspected that the great penetration in Mr. R.'s experiment was owing to the splitting of his timber in some degree.

PROBLEM VIII.

A 24-pounder ball being fired into a bank of firm earth, with a velocity of 1300 *feet per second, penetrated* 15 *feet. It is required then to ascertain the comparative resistances of elm and earth.*

COMPARING the numbers here with those in prob. 5, it is

is $\frac{f}{F} = \frac{dv^2s}{DV^2s} = \frac{2 \times 1500^2 \times 15 \times 12}{5 \cdot 55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^3 \times 0 \cdot 37} = \frac{1800}{271} = \frac{20}{3}$ nearly $= 6\frac{2}{3}$ nearly. That is, elm timber resists about $6\frac{2}{3}$ times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm; supposing it fired with a velocity of 1700 feet per second, and that the lead does not change its figure by the stroke against the wood.

HERE $D = \frac{3}{4}$, $N = 11\frac{1}{3}$, $n = 7\frac{1}{3}$. Then, by the numbers and theorem in prob. 5, it is $S =$ - - - - - - - - - -
$\frac{DNV^2s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600}$
$= 9\frac{2}{3}$ inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber must have resisted about twice as much; or else, which is much more probable, that the defect in his penetration arose from the change of figure in the leaden ball he used, from the blow against the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 feet per second, having been found to penetrate 13 inches deep into dry elm: It is required to ascertain the time of passing through every single inch of the 13, and the velocity lost at each of them; supposing the resistance of the wood constant or uniform.

THE velocity v being 1500 feet, or $1500 \times 12 = 18000$ inches, and velocities and times being as the roots of the spaces, in constant retarding forces, as well as in accelerating ones, and t being $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$ part of a second, the whole time of passing through the 13 inches; therefore as

$\sqrt{1}$

$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

veloc. lost		Time in the	
$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}v = 58\cdot9 :: t :$	$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}}t =$	$\cdot00005$	1st [inch.
$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}v = 61\cdot4 :: t :$	$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}}t =$	$\cdot00006$	2d
$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}v = 64\cdot2$ &c	$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}}t =$	$\cdot00006$	3d
$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}v = 67\cdot5$	$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}}t =$	$\cdot00007$	4th
$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}v = 71\cdot4$	$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}}t =$	$\cdot00007$	5th
$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}v = 76\cdot0$	$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}}t =$	$\cdot00007$	6th
$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}v = 81\cdot7$	$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}}t =$	$\cdot00008$	7th
$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}v = 88\cdot8$	$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}}t =$	$\cdot00008$	8th
$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}v = 98\cdot2$	$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}}t =$	$\cdot00009$	9th
$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}v = 111\cdot4$	$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}}t =$	$\cdot00011$	10th
$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}v = 132\cdot2$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}}t =$	$\cdot00013$	11th
$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}v = 172\cdot3$	$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}}t =$	$\cdot00017$	12th
$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}v = 416\cdot0$	$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}}t =$	$\cdot00040$	13th
Sum $1500\cdot0$	Sum $\frac{1}{692}$ or	$\cdot00144$	sec.

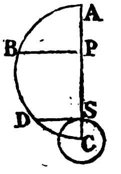
Hence, as the motion lost at the beginning is very small; and consequently the motion communicated to any body, as an inch plank, in passing through it, is very small also; we can conceive how such a plank may be shot through, when standing upright, without oversetting it.

PROBLEM

PROBLEM XI.

The force of attraction, above the earth, being inversely as the square of the distance from the centre; it is proposed to determine the time, velocity, and other circumstances, attending a heavy body falling from any given height; the descent at the earth's surface being $16\frac{1}{12}$ feet, or 193 inches, in the first second of time.

Put



r = CS the radius of the earth,
a = CA the dist. fallen from,
x = CP any variable distance,
v = the velocity at P,
t = time of falling there, and
g = $16\frac{1}{12}$, half the veloc. or force at S,
f = the force at the point P.

Then we have the three following equations, viz.

$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity is considered as 1;

$\dot{t}v = -\dot{x}$, because x decreases; and

$v\dot{v} = -2gf\dot{x} = -\frac{2gr^2\dot{x}}{x^2}$.

The fluents of the last equation give $v^2 = \frac{4gr^2}{x}$. But when $x = a$, the velocity $v = 0$; therefore, by correction, $v^2 = \frac{4gr^2}{x} - \frac{4gr^2}{a} = 4gr^2 \times \frac{a-x}{ax}$; or $v = \sqrt{(\frac{4gr^2}{a} \times \frac{a-x}{x})}$, a general expression for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{(4gr \times \frac{a-r}{a})}$ for the greatest velocity, or the velocity when the body strikes the earth.

When a is very great in respect of r, the last velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{4gr}$ very nearly, or nearly $\sqrt{4gr}$ only, which is accurately the greatest velocity by falling from an infinite height. And this, when $r = 3965$ miles, is 6·9506 miles per second. Also, the velocity acquired in falling from the

the distance of the sun, or 12000 diameters of the earth, is 6·9505 miles per second. And the velocity acquired in falling from the distance of the moon, or 30 diameters, is 6·8927 miles per second.

Again, to find the time; since $t\dot{v} = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax - xx}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax - xx} +$ arc to diameter a and vers. $a - x$); or the time of falling to any point P $= \frac{1}{2r}\sqrt{\frac{a}{g}} \times$ (AB + BP). And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{g}} \times \frac{\text{AD} + \text{DS}}{\text{SC}}$ for the whole time of falling to the surface at S; which is evidently infinite when a or AC is infinite, though the velocity is then only the finite quantity $\sqrt{4gr}$.

When the height above the earth's surface is given $= g$; because r is then nearly $= a$, and AD nearly $=$ DS, the time t for the distance g will be nearly - - - - - - - - - - $\sqrt{\frac{1}{4gr^2}} \times 2\text{DS} = \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1''$, as it ought to be.

If a body, at the distance of the moon at A, fall to the earth's surface at S: Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$ da. 19 h. 46′ 46″, which is the time of falling from the moon to the earth.

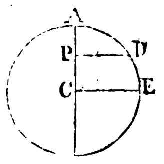
When the attracting body is considered as a point C; the whole time of descending to C will be - - - - - - - $\frac{1}{2r}\sqrt{\frac{a}{g}} \times \text{ABDC} = \frac{\cdot 7854a}{r}\sqrt{\frac{a}{g}} = \frac{10a}{51r}\sqrt{a} = \frac{\cdot 7854}{r}\sqrt{\frac{a^3}{g}}$.

Hence, the times employed by bodies, in falling from quiescence to the centre of attraction, are as the square roots of the cubes of the heights from which they respectively fall.

PROBLEM XII.

The force of attraction below the earth's surface being directly as the distance from the centre: it is proposed to determine the circumstances of velocity, time, and space fallen by a heavy body from the surface, through a perforation made straight to the centre of the earth: abstracting from the effect of the earth's rotation, and supposing it to be a homogeneous sphere of 3965 miles radius.

Put



Put r = AC the radius of the earth,
x = CP the dist. from the centre,
v = the velocity at P,
t = the time there,
g = $16\frac{1}{12}$, half the force at A,
f = the force at P.

Then CA : CP :: 1 : f; and the three equations are $rf = x$, and $v\dot{v} = -2gf\dot{x}$, and $t\dot{v} = -\dot{x}$.

Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-2gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{(2g \times \frac{r^2 - x^2}{r})} = \text{PD}\sqrt{\frac{2g}{r}} = \text{PD}\sqrt{\frac{2g}{\text{CE}}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular or sine PD at that point.

When the body arrives at C, then $v = \sqrt{2gr} = \sqrt{2g \cdot \text{AC}}$ = 25950 feet or 4·9148 miles per second, which is the greatest velocity, or that at the centre C.

Again, for the time; $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$; and the fluents give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{r} = \sqrt{\frac{1}{2gr}} \times$ arc AD. So that the time of descent to any point P, is as the corresponding arc AD.

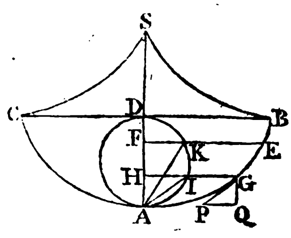
When P arrives at C, the above becomes $t = \sqrt{\frac{1}{2gr}} \times$ quadrant AE $= \frac{\text{AE}}{\text{AC}}\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$ seconds = 21′ 7″$\frac{1}{4}$, for the time of falling to the centre C.

The time of falling to the centre is the same quantity $1{\cdot}5708\sqrt{\frac{r}{2g}}$, from whatever point in the radius AC the body begins to move. For, let n be any given distance from C at which the motion commences: then by correction, $v = \sqrt{(\frac{2g}{r} \cdot \overline{n^2 - x^2})}$, and hence $\dot{t} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}}$, the fluents of which give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{2g}} \times$ quadrant $= 1{\cdot}5708\sqrt{\frac{r}{2g}}$, for the time of descent to the centre C, the same as before.

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A. Then from B it will return again in the same manner, through C to A; and so oscillate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or $= 2 \times 3{\cdot}1416\sqrt{\frac{r}{2g}} =$ 1h. 24′ 29″.

PROBLEM XIII.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.



Let

S be the point of suspension;

SA, the length of pendulum;

CAB, the whole cycloidal arc;

AIKD, the generating circle, to which FKE, HIG are perpendiculars.

SC, SB two other equal semicloids, on which the thread wrapping, the end A is made to describe the cycloid BAC.

Now, by the nature of the cycloid, AD = DS; and SA = 2AD = SC = SB = SA = AB. Also, if at any point G be drawn the tangent GP; also GQ parallel and PQ perpendicular to AD. Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direction GP : : GP : GQ : : AI : AH : : AD : AI; in like manner, the force of gravity : force in the curve at E : : AD : AK; that is, the accelerative force in the curve, is every where as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always = 2AI, by the nature of the curve. So that the process and conclusions, for the velocity and time of describing any arc in this case, will be the very same as in the last problem, the nature of the forces being the same, viz. as the distance to be passed over to the lowest point A.

From

From which it follows, that the time of a semi-vibration, in all arcs, AG, AE, &c, is the same constant quantity $1{\cdot}5708\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{AS}{2g}} = 1{\cdot}5708\sqrt{\frac{l}{2g}}$; and the time of a whole vibration from B to C, or from C to B, is $3{\cdot}1416\sqrt{\frac{l}{2g}}$; where $l = AS = AB$ is the length of the pendulum, $g = 16\frac{1}{12}$ feet or 193 inches, and 3·1416 the circumference of a circle whose diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, by the nature of descents, is $\sqrt{\frac{l}{2g}}$, which being in proportion to $3{\cdot}1416\sqrt{\frac{l}{2g}}$, as 1 is to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, is to the time of one vibration.

If the time of the whole vibration be 1 second, this equation arises, viz. $1'' = 3{\cdot}1416\sqrt{\frac{l}{2g}}$; hence $l = \frac{2g}{3{\cdot}1416^2} = \frac{g}{4{\cdot}9348}$, and $g = 3{\cdot}1416^2 \times \frac{1}{2}l = 4{\cdot}9348l$. So that if one of these, g or l, be given by experiment, these equations will give the other. When g, for instance, is supposed to be given $= 16\frac{1}{12}$ feet, or 193 inches; then is $l = \frac{g}{4{\cdot}9348} = 39{\cdot}11$, the length of a pendulum to vibrate seconds. Or if $l = 39\frac{1}{8}$, the length of the seconds pendulum for the latitude of London, by experiment; then is $g = 4{\cdot}9348l = 193.07$ inches $= 16\frac{107}{1200}$ feet, or nearly $16\frac{1}{12}$ feet, for the space descended by gravity in the first second of time in the latitude of London; also agreeing with experiment.

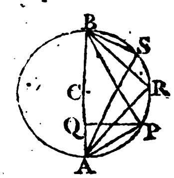
Hence the times of vibration of pendulums, are as the square roots of their lengths; and the number of vibrations made in a given time, is reciprocally as the square roots of the lengths. And hence also, the length of a pendulum vibrating n times in a minute, or 60″, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nn}$.

When a pendulum vibrates in a circular arc; as the length of the string is constantly the same, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is smaller; so that

when it is very small, the times of vibration will be nearly equal. And hence it happens that $39\frac{1}{8}$ inches is the length of a pendulum vibrating seconds, in the very small arc of a circle.

PROBLEM XIV.

To determine the Time of a Body descending down the Chord of a Circle.



Let C be the centre; AB the vertical diameter; AP any chord, down which a body is to descend from P to A; and PQ perpendicular to AB.

Now, as the natural force of gravity in the vertical direction BA, is to the force urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular distances below PQ; and consequently the time in PA : time in QA : : PA : QA : : BA : PA; but time in BA : time in QA : : $\sqrt{BA}$: $\sqrt{QA}$: : BA : PA; hence, as three of the terms in each proportion are the same, the fourth terms must be equal, namely the time in BA = the time PA.

And, in like manner, the time in BP = the time in BA. So that, in general, the times of descending down all the chords BA, BP, BR, BS, &c, or PA, RA, SA, &c, are all equal, and each equal to the time of falling freely through the diameter; as before found at art. 131, Mechanics. Which time is $\sqrt{\frac{2r}{g}}$, where $g = 16\frac{1}{12}$ feet, and r = the radius AC; for $\sqrt{g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{g}}$.

PROBLEM XV.

To determine the Time of filling the Ditches of a Work with Water, at the Top, by a Sluice of 2 Feet square; the Head of Water above the Sluice being 10 Feet, and the Dimensions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.

The capacity of the ditch is 189000 cubic feet.

But $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10g}$ the velocity of the water through the sluice, the area of which is 4 square feet; therefore

therefore $8\sqrt{10g}$ is the quantity per second running through it; and consequently $8\sqrt{10g} : 189000 :: 1'' : \frac{23625}{\sqrt{10g}} = 1863''$ or $31' 3''$ nearly, which is the time of filling the ditch.

PROBLEM XVI.

To determine the Time of emptying a Vessel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom: the Height of the Aperture being very small in respect of the Altitude of the Fluid.

PUT a = the area of the aperture or sluice;
$2g = 32\frac{1}{6}$ feet, the force of gravity;
d = the whole depth of water;
x = the variable altitude of the surface above the aperture;
A = the area of the surface of the water.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the fluid will issue at the sluice; and hence $A : a :: 2\sqrt{gx} : \frac{2a\sqrt{gx}}{A}$, the velocity with which the surface of the water will descend at the altitude x, or the space it would descend in 1 second with the velocity there. Now, in descending the space $\dot{x}$, the velocity may be considered as uniform; and uniform descents are as their times; therefore $\frac{2a\sqrt{gx}}{A} : \dot{x} :: 1'' : \frac{A\dot{x}}{2a\sqrt{gx}}$ the time of descending $\dot{x}$ space, or the fluxion of the time of exhausting. That is, $\dot{t} = \frac{-A\dot{x}}{2a\sqrt{gx}}$; which is made negative, because x is a decreasing quantity, or its fluxion negative.

Now, when the nature or figure of the vessel is given, the area A will be given in terms of x; which value of A being substituted into this fluxion of the time, the fluent of the result will be the time of exhausting sought.

So if, for example, the vessel be any prism, or everywhere of the same breadth; then A is a constant quantity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{g}}$, and should be 0; therefore the correct fluent is $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{g}}$ for the time of the surface de-

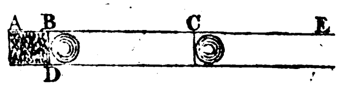
scending till the depth of the water be x. And when $x = 0$, the whole time of exhausting is barely $\frac{A}{a}\sqrt{\frac{d}{g}}$.

Hence, if A be = 10000 square feet, $a = 1$ square foot, and $d = 10$ feet; the time is $7885\frac{1}{5}$ seconds, or 2h. 11′ $25''\frac{1}{5}$.

Again, if the vessel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $90 : 90 + x :: 20 : \frac{90 + x}{9} \times 2$ the breadth of the surface of the water when its depth in the canal is x; and therefore $A = \frac{90 + x}{9} \times 2000$ is the surface at that time. Consequently $\dot{t}$ or $\frac{-A\dot{x}}{2a\sqrt{gx}} = 1100 \times \frac{90 + x}{9} \times \frac{-\dot{x}}{a\sqrt{gx}}$ is the fluxion of the time; the correct fluent of which, when $x = 0$, is $1000 \times \frac{180 + \frac{2}{3}d}{9a} \times \sqrt{\frac{d}{g}} = \frac{1000 \times 186 \times 3}{9 \times 4\frac{1}{96}} =$ $15459''\frac{2}{3}$ nearly, or 4h. 17′ $39''\frac{2}{3}$, being the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XVII.

To determine the Velocity with which a Ball is discharged from a Given Piece of Ordnance, with a Given Charge of Gunpowder.



Let the annexed figure represent the bore of the gun; AD being the part filled with gunpowder.

And put

a = AB, the part at first filled with powder and the bag;
b = AE, the whole length of the gunbore;
c = ·7854, the area of a circle whose diameter is 1;
d = BD, the diameter of the ball;
e = the specific gravity of the ball, or weight of 1 cubic foot;
g = $16\frac{1}{12}$ feet, descended by a body in 1 second;
m = 230 ounces, the pressure of the atmosphere on a sq. inch;
n to 1 the ratio of the first force of the fired powder, to the pressure of the atmosphere;
w = the weight of the ball. Also, let
x = AC, be any variable distance of the ball from A, in moving along the gunbarrel.

First,

First, cd^2 is $=$ the area of the circle BD of the ball; there. mcd^2 is the pressure of the atmosphere on BD; conseq. $mncd^2$ is the first force of the powder on BD.

But the force of the inflamed powder is proportional to its density, and the density is inversely as the space it fills; therefore the force of the powder on the ball at B, is to the force on the same at C, as AC is to AB; that is, - - - -

$x : a :: mncd^2 : \frac{mnacd^2}{x} = F$, the motive force at C:

conseq. $\frac{F}{w} = \frac{mnacd^2}{wx} = f$, the accelerating force there.

Hence, theor. 10 of forces gives $v\dot{v} = 2gf\dot{x} = \frac{2gmnacd^2}{w} \times \frac{\dot{x}}{x}$;

the fluent of which is $v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. of x.

But when $v = 0$, then $x = a$; theref. by correction,

$v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. $\frac{x}{a}$ is the correct fluent; conseq.

$v = \sqrt{(\frac{4gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a})}$ is the vel. of the ball at C.

and $v = \sqrt{(\frac{4gmnbcd^2}{w} \times \text{hyp. log. } \frac{b}{a})}$ the velocity with which the ball issues from the muzzle at E; where b denotes the length of the cylinder filled with powder; and a the length to the hinder part of the ball, which will be more than b when the powder does not touch the ball.

Or, by substituting the numbers for g and m, and changing the hyperbolic logarithms for the common ones, then

$v = \sqrt{(\frac{2230nbd^2}{w} \times \text{com. log. } \frac{b}{a})}$, the velocity at E, in feet.

But, the content of the ball being $\frac{2}{3}cd^3$, its weight is -

$w = \frac{\frac{2}{3}cd^3e}{12^3} = \frac{ced^3}{2592} = \frac{ed^3}{3300}$; which being substituted for w, in the value of v, it becomes

$v = 2713\sqrt{(\frac{nb}{de} \times \text{com. log. } \frac{b}{a})}$, the velocity at E.

When the ball is of cast iron; taking $e = 7368$, the rule becomes

$v = 100\sqrt{(\frac{nb}{10d} \times \log. \frac{b}{a})}$ for the veloc. of the cast-iron ball.

Or, when the ball is of lead; then - - - - - - - -

$v = 80\frac{2}{3}\sqrt{(\frac{nb}{10d} \times \log. \frac{b}{a})}$ for the veloc. of the leaden ball.

Corol. From the general expression for the velocity v, above given, may be derived what must be the length of the charge of powder a, in the gun-barrel, so as to produce the greatest possible velocity in the ball; namely, by making the value of v a maximum, or, by squaring and omitting the constant quantities, the expression $a \times$ hyp. log. of $\frac{b}{a}$ a maximum, or its fluxion equal to nothing; that is $\dot{a} \times$ hyp. log. $\frac{b}{a} - \dot{a} = 0$, or hyp. log. of $\frac{b}{a} = 1$; hence $\frac{b}{a} = 2{\cdot}71828$, the number whose hyp. log. is 1. So that $a : b :: 1 : 2{\cdot}71828$, or as 4 to 11 nearly, or nearer as 7 to 19; that is, the length of the charge, to produce the greatest velocity, is the $\frac{4}{11}$th part of the length of the bore, or nearer $\frac{7}{19}$ of it.

By actual experiment it is found, that the charge for the greatest velocity, is but little less than that which is here computed from theory; as may be seen by turning to page 269 of my volume of Tracts, where the corresponding parts are found to be, for four different lengths of gun, thus, $\frac{3}{10}$, $\frac{3}{12}$, $\frac{3}{16}$, $\frac{3}{20}$; the parts here varying, as the gun is longer, which allows time for the greater quantity of powder to be fired, before the ball is out of the bore.

SCHOLIUM.

In the calculation of the foregoing problem, the value of the constant quantity n remains to be determined. It denotes the first strength or force of the fired gunpowder, just before the ball is moved out of its place. This value is assumed, by Mr. Robins, equal to 1000, that is, 1000 times the pressure of the atmosphere, on any equal spaces.

But the value of the quantity n may be derived much more accurately, from the experiments related in my Tracts, by comparing the velocities there found by experiment, with the rule for the value of v, or the velocity, as above computed by theory, viz. - - - - - - - - - - - - -

$$v = 100\sqrt{(\frac{na}{10d} \times \text{log. of } \frac{b}{a})}, \text{ or } = 100\sqrt{(\frac{nb}{10d} \times \text{log. of } \frac{b}{a})}.$$

Now, supposing that v is a given quantity, as well as all the other quantities, excepting only the number n, then by reducing this equation, the value of the letter n is found to be as follows, viz. - - - - - - - - - - - - - - - - -

$$n = \frac{dvv}{1000a} \div \text{com. log. of } \frac{b}{a}, \text{ or } = \frac{dvv}{1000b} \div \text{log. of } \frac{b}{a},$$

when b is different from a.

Now,

Now, to apply this to the experiments. By page 257 of the Tracts, the velocity of the ball, of 1·96 inches diameter, with 4 ounces of powder, in the gun No. 1, was 1100 feet per second; and, by page 109, the length of the gun, when corrected for the spheroidal hollow in the bottom of the bore, was 28·53; also, by page 237, the length of the charge, when corrected in like manner, was 3·45 inches of powder and bag together, but 2·54 of powder only: so that the values of the quantities in the rule, are thus: $a = 3\cdot45$; $b = 28\cdot53$; $d = 1\cdot96$; $h = 2\cdot54$; and $v = 1100$: then, by substituting these values instead of the letters, in the theorem $n = \frac{dvv}{1000a} \div$ com. log. of $\frac{b}{a}$, it comes out $n = 750$, when h is considered as the same as a. And so on, for the other experiments there treated of.

It is here to be noted however, that there is a circumstance in the experiments delivered in the Tracts, just mentioned, which will alter the value of the letter a in this theorem, which is this, viz. that a denotes the distance of the shot from the bottom of the bore; and the length of the charge of powder alone ought to be the same thing; but, in the experiments, that length included, besides the length of real powder, the substance of the thin flannel bag in which it was always contained, of which the neck at least extended a considerable length, being the part where the open end was wrapped and tied close round with a thread. This circumstance causes the value of n, as found by the theorem above, to come out less than it ought to be, for it shows the strength of the inflamed powder when just fired, and when the flame fills the whole space a before occupied both by the real powder and the bag, whereas it ought to show the first strength of the flame when it is supposed to be contained in the space only occupied by the powder alone, without the bag. The formula will therefore bring out the value of n too little, in proportion as the real space filled by the powder is less than the space filled both by the powder and its bag. In the same proportion therefore must we increase the formula, that is, in the proportion of h, the length of real powder, to a the length of powder and bag together. When the theorem is so corrected, it becomes $\frac{dvv}{1000h} \div$ com. log. of $\frac{b}{a}$.

Now, by pa. 237 of the Tracts, there are given both the lengths of all the charges, or values of a, including the bag, and also the length of the neck and bottom of the bag, which is 0·91 of an inch, which therefore must be subtracted from

C... ...esponding values of ... above ... ·45 to 2·54. ... charg... ...e result 750, in proportion grea... ... And so on for the other val...

cor... ...ge the results in a table, with ...en corrected, from which they are ...

...als and First Force of Powder, &c.*

Length of the gun	Charge of Powder.			Velocity or value of v.	First force, or value of n.
	Weight in ounces	Length or value of a.	Length or value of h.		
Inches.	4	3·45	2·54	1100	1018
.5 53	8	5·99	5·08	1430	1164
	16	11·07.	10·16	1430	967
	4	3·45	2·54	1180	1077
38·43	8	5·99	5·08	1580	1193
	16	11·07	10·16	1660	984
	4	3·45	2·54	1300	1067
57·70	8	5·99	5·08	1790	1256
	16	11·07	10·16	2000	1076
	4	3·45	2·54	1370	1060
80·23	8	5·99	5·08	1940	1289
	16	11·07	10·16	2200	1085

Where it may be observed, that the numbers in the column ... velocities, 1430 and 2200, are a little increased, as, from ... view of the table of experiments, they evidently required ... be. Also the value of the letter d is constantly 1·96 ...

Hence it appears, that the value of the letter n, used in ... theorem, though not yet greatly different from the num... 1000, assumed by Mr. Robins, is rather various, both ... the different lengths of the gun, and for the different ...ges with the same gun.

But

But this diversity in the value of the quantity n, or the first force of the inflamed gunpowder, is probably owing in some measure to the omission of a material datum in the calculation of the problem, namely, the weight of the charge of powder, which has not at all been brought into the computation. For it is manifest, that the elastic fluid has not only the ball to move and impel before it, but its own weight of matter also. The computation may therefore be renewed, in the ensuing problem, to take that datum into the account.

PROBLEM XVIII.

To determine the same as in the last Problem; taking both the Weight of Powder and the Ball into the Calculation.

BESIDES the notation used in the last problem, let $2p$ denote the weight of the powder in the charge, with the flannel bag in which it was inclosed.

Now, because the inflamed powder occupies at all times the part of the gun bore which is behind the ball, its centre of gravity, or the middle part of the same, will move with only half the velocity that the ball moves with; and this will require the same force as half the weight of the powder, &c, moved with the whole velocity of the ball. Therefore, in the conclusion derived in the last problem, we are now, instead of w, to substitute the quantity $p + w$; and when that is done, the last velocity will come out, $v = \sqrt{(\frac{2230nbd^2}{p + w} \times \text{com. log. } \frac{b}{a})}$.

And from this equation is found the value of n, which is $n = \frac{p + w}{2230bd^2} v^2 \div \text{log. of } \frac{b}{a}, = \frac{p + w}{8567b} v^2 \div \text{log. of } \frac{b}{a}$, by substituting for d its value 1·96, the diameter of the ball.

Now as to the ball, its medium weight was 16 oz. 13 dr. = 16·81 oz. And the weights of the bags containing the several charges of powder, viz. 4 oz, 8 oz, 16 oz, were 8 dr, 12 dr, and 1 oz. 5 dr; then, adding these to the respective contained weights of powder, the sums, 4·5 oz, 8·75 oz, 17·31 oz, are the values of $2p$, or the weights of the powder and bags; the halves of which, or 2·25, and 4·38, and 8·66, are the values of the quantity p for those three charges; and these being added to 16·81, the constant weight of the ball, there are obtained the three values of $p + w$ for the three charges of powder, which values therefore are 19·06 oz, and 21·19 oz, and 25·47 oz. Then, by calculating the values of the first force n, by the last rule above, with these new data, the whole will be found as in the following table.

The

The Gun.		Charge of Powder.			Weight of ball and charge, or values of $p + w$	Velocity, or the values of v.	First force or the value of n.
No.	Length or value of b.	Weight in ounces.	Length or value of a.	Length or value of h.			
	inches	4	3·45	2·54	19·06	1100	1155
1	28·53	8	5·99	5·08	21·19	1430	1470
		16	11·07	10·16	25·47	1430	1456
		4	3.45	2·54	19·06	1180	1167
2	38·43	8	5·99	5·08	21·19	1580	1506
		16	11·07	10·16	25·47	1660	1492
		4	3·45	2·54	19·06	1300	1210
3	57·70	8	5·99	5·08	21·19	1790	1586
		16	11·07	10·16	25·47	2000	1646
		4	3·45	2·54	19·06	1370	1203
4	80·23	8	5·99	5·08	21·19	1940	1627
		16	11·07	11·16	25.47	2200	1648

And here it appears that the values of n, the first force of the charge, are much more uniform and regular than by the former calculations in the preceding problem, at least in all excepting the smallest charge, 4 oz, in each gun; which it would seem must be owing to some general cause or causes. Nor have we long to search, to find out what those causes may be. For when it is considered that these numbers for the value of n, in the last column of the table, ought to exhibit the first force of the fired powder, when it is supposed to occupy the space only in which the bare powder itself lies; and that whereas it is manifest that the condensed fluid of the charge in these experiments, occupies the whole space between the ball and the bottom of the gun bore, or the whole space taken up by the powder and the bag or cartridge together, which exceeds the former space, or that of the powder alone, at least in the proportion of the circle of the gun bore, to the same as diminished by the thickness of the surrounding flannel of the bag that contained the powder; it is manifest that the force was diminished on that account. Now by gently compressing a number of folds of the flannel together, it has been found that the thickness of the single flannel was equal to the 40th part of an inch; the double of which, $\frac{1}{20}$ or ·05 of an inch, is therefore the quantity

quantity by which the diameter of the circle of the powder within the bag, was less than that of the gun bore. But the diameter of the gun bores was 2·02 inches; therefore, deducting the ·05, the remainder 1·97 is the diameter of the powder cylinder within the bag: and because the areas of circles are to each other as the spaces of their diameters, and the squares of these numbers, 1·97 and 2·02, being to each other as 388 to 408, or as 97 to 102; therefore, on this account alone, the numbers before found, for the value of *n*, must be increased in the ratio of 97 to 102.

But there is yet another circumstance, which occasions the space at first occupied by the inflamed powder to be larger than that at which it has been taken in the foregoing calculations, and that is the difference between the content of a sphere and cylinder. For the space supposed to be occupied at first by the elastic fluid, was considered as the length of a cylinder measured to the hinder part of the curve surface of the ball, which is manifestly too little by the difference between the content of half the ball and a cylinder of the same length and diameter, that is, by a cylinder whose length is $\frac{1}{3}$ the semidiameter of the ball. Now that diameter was 1·96 inches; the half of which is 0·98, and $\frac{1}{3}$ of this is 0·33 nearly. Hence then it appears that the lengths of the cylinders, at first filled by the dense fluid, viz. 3·45, and 5·99, and 11·07, have been all taken too little by 0·33; and hence it follows that, on this account also, all the numbers before found for the value of the first force *n*, must be further increased in the ratios of 3·45 and 5·99 and 11·07, to the same numbers increased by 0·33, that is, to the numbers 3·78 and 6·32 and 11·40.

Compounding now these last ratios with the foregoing one, viz. 97 to 102, it produces these three, viz. the ratios of 334 and 581 and 1074, respectively to 385 and 647 and 1163. Therefore increasing the last column of numbers, for the value of *n*, viz. those of the 4 oz. charge in the ratio of 334 to 385, and those of the 8 oz. charge in the ratio of 581 to 647, and those of the 16 oz. charge in the ratio of 1074 to 1163, with every gun, they will be reduced to the numbers in the annexed table; where the numbers are still larger and more regular than before.

Powder.	The Guns.			
oz.	1	2	3	4
4	1372	1387	1438	1480
8	1637	1677	1766	1812
16	1577	1616	1782	1784

Thus

Thus then at length it appears that the first force of the inflamed gunpowder, when occupying only the space at first filled with the powder, is about 1800, that is 1800 times the elasticity of the natural air, or pressure of the atmosphere, in the charges with 8 oz. and 16 oz. of powder, in the two longer guns; but somewhat less in the two shorter, probably owing to the gradual firing of gunpowder in some degree; and also less in the lowest charge 4 oz, in all the guns, which may probably be owing to the less degree of heat in the small charge. But besides the foregoing circumstances that have been noticed, or used in the calculations, there are yet several others that might and ought to be taken into the account, in order to a strict and perfect solution of the problem; such as, the counter pressure of the atmosphere, and the resistance of the air on the fore part of the ball while moving along the bore of the gun; the loss of the elastic fluid by the vent and windage of the gun; the gradual firing of the powder; the unequal density of the elastic fluid in the different parts of the space it occupies between the ball and the bottom of the bore; the difference between pressure and percussion when the ball is not laid close to the powder; and perhaps some others: on all which accounts, it is probable that, instead of 1800, the first force of the elastic fluid is not less than 2000 times the strength of natural air.

Corol. From the theorem last used for the velocity of the ball and elastic fluid, viz. $v = \sqrt{(\frac{2230bd^2}{p+w}n \div \log. \frac{b}{a})} = \sqrt{\frac{8567bn}{p+w} \div \log. \frac{b}{a}})$, we may find the velocity of the elastic fluid alone, viz. by taking w, or the weight of the ball, $= 0$ in the theorem, by which it becomes barely $v = \sqrt{(\frac{8567bn}{p} \div \log. \frac{b}{a})}$, for that velocity. And by computing the several preceding examples by this theorem, supposing the value of n to be 2000, the conclusions come out a little various, being between 4000 and 5000, but most of them nearer to the latter number. So that it may be concluded that the velocity of the flame, or of the fired gun-powder, expands itself at the muzzle of the gun, at the rate of about 5000 feet per second nearly.

On

On the MOTION of BODIES in FLUIDS.

PROBLEM XIX.

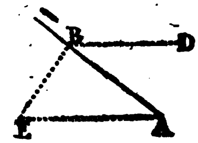
To determine the Force of Fluids in Motion; and the Circumstances attending Bodies Moving in Fluids.

1. It is evident that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid on the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion, must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the specific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{4g}$, the whole resistance, or motive force m, will be $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$; g being $16\frac{1}{12}$ feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

2. This ratio, of the square of the velocity, may be otherwise derived thus. The force of the fluid in motion, must be as the force of one particle multiplied by the number of them; but the force of a particle is as its velocity; and the number of them striking the plane in a given time, is also as the velocity; therefore the whole force is as $v \times v$ or v^2, that is, as the square of the velocity.

3. If the direction of motion, instead of being perpendicular to the plane, as above supposed, be inclined to it in any angle, the sine of that angle being s, to the radius 1; then the resistance to the plane, or the force of the fluid against

against the plane, in the direction of the motion, as assigned above, will be diminished in the triplicate ratio of radius to the sine of the angle of inclination, or in the ratio of 1 to s^3. For, AB being the direction of the plane, and BD that of the motion, making the angle ABD, whose sine is s; the number of particles, or quantity of the fluid striking the plane, will be diminished in the ratio of 1 to s, or of radius to the sine of the angle B of inclination; and the force of each particle will also be diminished in the same ratio of 1 to s: so that, on both these accounts, the whole resistance will be diminished in the ratio of 1 to s^2, or in the duplicate ratio of radius to the sine of the said angle. But again, it is to be considered that this whole resitance is exerted in the direction BE perpendicular to the plane; and any force in the direction BE, is to its effect in the direction AE, parallel to BD, as AE to BE, that is as 1 to s. So that finally, on all these accounts, the resistance in the direction of motion, is diminished in the ratio of 1 to s^3, or in the triplicate ratio of radius to the sine of inclination. Hence, comparing this with article 1, the whole resistance, or the motive force on the plane, will be $m = \frac{anv^2s^3}{4g}$.



4. Also, if w denote the weight of the body, whose plane face a is resisted by the absolute force m; then the retarding force f, or $\frac{m}{w}$, will be $\frac{anv^2s^3}{4gw}$.

5. And if the body be a cylinder, whose face or end is a, and diameter d, or radius r, moving in the direction of its axis; because then $s = 1$, and $a = pr^2 = \frac{1}{4}pd^2$, where $p = 3{\cdot}1416$; the resisting force m will be - - - - - - $\frac{npd^2v^2}{16g} = \frac{npr^2v^2}{4g}$, and the retarding force $f = \frac{npd^2v^2}{16gw} = \frac{npr^2v^2}{4gw}$.

6. This is the value of the resistance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical surface, or an elliptic section, or any other figure every where equally inclined to the axis, the sine of inclination being s: then the number of particles of the fluid striking the face being still the same, but the force of each, opposed to the direction

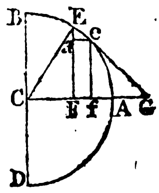
of

of motion, diminished in the duplicate ratio of radius to the sine of inclination, the resisting force m would be $\frac{npd^2v^2s^4}{16g} = \frac{npr^2v^2s^2}{4g}$.

But if the body were terminated by an end or face of any other form, as a spherical one, or such like, where every part of it has a different inclination to the axis; then a further investigation becomes necessary, such as in the following proposition.

PROBLEM XX.

To determine the Resistance of a Fluid to any Body, moving in it, of a Curved End; as a Sphere, or a Cylinder with a Hemispherical End, &c.



1. LET BEAD be a section through the axis CA of the solid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: also, draw the perpendicular ordinates EF, ef, indefinitely near each other; and draw ae parallel to CG.

Putting CF $= x$, EF $= y$, BE $= z$, $s =$ sine $\angle$ G to radius 1, and $p = 3\cdot1416$: then $2py$ is the circumference whose radius is EF, or the circumference described by the point E, in revolving about the axis CA; and $2py \times$ Ee or $2py\dot{z}$ is the fluxion of the surface, or it is the surface described by Ee, in the said revolution about CA, and which is the quantity represented by a in art. 3 of the last problem: hence $\frac{nv^2s^3}{4g} \times 2py\dot{z}$ or $\frac{pnv^2s^3}{2g} \times y\dot{z}$ is the resistance on that ring, or the fluxion of the resistance to the body, whatever the figure of it may be. And the fluent of which will be the resistance required.

2. In the case of a spherical form; putting the radius CA or CB $= r$, we have $y = \sqrt{r^2 - x^2}$, $s = \frac{\text{EF}}{\text{EG}} = \frac{\text{CF}}{\text{CE}} = \frac{x}{r}$, and $y\dot{z}$, or EF $\times$ Ee $=$ CE $\times$ ae $= r\dot{x}$; therefore the general fluxion $\frac{pnv^2}{2g} \times s^3y\dot{z}$ becomes $\frac{pnv^2}{2g} \times \frac{x^3}{r^3} \times r\dot{x} = \frac{pnv^2}{2gr^2} \times x^3\dot{x}$;

the

the fluent of which, or $\frac{pnv^2}{8gr^2}x^4$, is the resistance to the spherical surface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{pnv^2r^2}{8g}$ for the resistance on the whole hemisphere; which is also equal to $\frac{pnv^2d^2}{32g}$, where $d = 2r$ the diameter.

3. But the perpendicular resistance to the circle of the same diameter d or BD, by art. 5 of the preceding problem, is $\frac{pnv^2d^2}{16g}$; which, being double the former, shows that the resistance to the sphere, is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.

4. Since $\frac{1}{6}pd^3$ is the magnitude of the globe; if N denote its density or specific gravity, its weight w will be $= \frac{1}{6}pd^3\text{N}$, and therefore the retardive force f or $\frac{m}{w} = \frac{pnv^2d^2}{32g} \times \frac{6}{p\text{N}d^3}$ $= \frac{3nv^2}{16g\text{N}d}$; which is also $= \frac{v^2}{4gs}$ by art. 8 of the general theorems in page 342; hence then $\frac{3n}{4\text{N}d} = \frac{1}{s}$, and $s = \frac{\text{N}}{n} \times \frac{4}{3}d$; which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by that accelerating or retarding force.

5. Hence the greatest velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the fluid, will be found by making the resisting force equal to that weight. For, after the velocity is arrived at such a degree, that the resisting force is equal to the weight that urges it, it will increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and n being the separate specific gravities of the globe and fluid, $\text{N} - n$ will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}pd^3\ (\text{N} - n)$ is the weight

weight by which it is urged; also $m =$ - - - - - - - $\frac{pnv^2d^2}{32g}$ is the resistance; consequently $\frac{pnv^2d^2}{32g} = \frac{1}{6}pd^3\,(\text{N} - n)$ when the velocity becomes uniform; from which equation is found $v = \sqrt{(4g \cdot \frac{4}{3}d \cdot \frac{\text{N} - n}{n})}$, for the said uniform or greatest velocity.

And, by comparing this form with that in art. 6 of the general theorems in page 342, it will appear that its greatest velocity, is equal to the velocity generated by the accelerating force $\frac{\text{N} - n}{n}$, in describing the space $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely describing the space $\frac{\text{N} - n}{n} \times \frac{4}{3}d$. If $\text{N} = 2n$, or the specific gravity of the globe be double that of the fluid, then $\frac{\text{N} - n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greatest velocity in describing $\frac{4}{3}d$ or $\frac{4}{3}$ of its diameter.—It is further evident, that if the body be very small, it will very soon acquire its greatest velocity, whatever its density may be.

EXAM. If a leaden ball, of 1 inch diameter, descend in water, and in air of the same density as at the earth's surface, the three specific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$. Then $v = \sqrt{4 \cdot 16\frac{1}{12} \cdot \frac{4}{36} \cdot 10\frac{1}{3}} = \frac{1}{9}\sqrt{31 \cdot 193} = 8{\cdot}5944$ feet, is the greatest velocity per second the ball can acquire by descending in water. And $v = \sqrt{4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3}}$ nearly $= \frac{50}{9}\sqrt{\frac{34{\cdot}193}{3}} = 259{\cdot}82$ is the greatest velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greatest velocities it could acquire, would be only $\frac{1}{10}$ of these, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were still further diminished, the greatest velocity would also be diminished, and that in the subduplicate ratio of the diameter of the ball.

PROBLEM XXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid, in which it is projected with a Given Velocity.

1. LET $a =$ the first velocity of projection, x the space described in any time t, and v the velocity then. Now, by art. 4 of the last problem, the accelerative force $f = \frac{3nv^2}{16g\text{N}d}$, where n is the density of the fluid, N that of the ball, and d its diameter. Therefore the general equation $v\dot{v} = 2gf\dot{s}$ becomes $v\dot{v} = - \frac{3nv^2}{8\text{N}d}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8\text{N}d}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8\text{N}d}$.

The correct fluent of this, is log. a − log. v or log. $\frac{a}{v} = bx$. Or, putting $c = 2{\cdot}718281828$, the number whose hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the first velocity, therefore the velocity lost in any time, will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx} - 1}{c^{bx}}$ part of the first velocity.

EXAMPLES.

EXAM. 1. If a globe be projected, with any velocity, in a medium of the same density with itself, and it describe a space equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8\text{N}d} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and $\frac{c^{bx} - 1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$ is the velocity lost, or nearly $\frac{2}{3}$ of the projectile velocity.

EXAM. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per second; to find the velocity lost after moving through any space, as suppose 500 feet of air: we should have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$, $x = 500$, $\text{N} = 7\frac{1}{2}$, $n = {\cdot}0012$; and therefore $bx =$ - - - $\frac{3nx}{8\text{N}d} = \frac{3 \cdot 12 \cdot 500 \cdot 3 \cdot 6}{8 \cdot 22 \cdot 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet per second: having lost 202 feet, or nearly $\frac{1}{6}$ of its first velocity.

EXAM. 3. If the earth revolved about the sun, in a medium as dense as the atmosphere near the earth's surface; and it were required to find the quantity of motion lost in a

year.

year. Then, since the earth's mean density is about $4\frac{1}{2}$, and its distance from the sun 12000 of its diameters, we have $24000 \times 3{\cdot}1416 = 75398$ diameters $= x$, and $bx = \frac{3 \,.\, 75398 \,.\, 12 \,.\, 2}{8 \,.\, 10000 \,.\, 9} = 7{\cdot}5398$; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{1880}{1881}$ parts are lost of the first motion in the space of a year, and only the $\frac{1}{1881}$ part remains.

EXAM. 4. If it be required to determine the distance moved, x, when the globe has lost any part of its motion, as suppose $\frac{1}{2}$, and the density of the globe and fluid equal; The general equation gives $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log.$ of $2 = 1{\cdot}8483925d$. So that the globe loses half its motion before it has described twice its diameter.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now, to find the fluent of this, put $z = c^{bx}$; then is $bx = \log. z$, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; conseq. $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} = \frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$; and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vanish together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}(\frac{1}{v} - \frac{1}{a})$ the time sought; where $b = \frac{3n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

EXAM. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per second; and it were required to determine in what time it would pass over 500 yards or 1500 feet, and what would be its velocity at the end of that time: We should have, as in exam. 2 above, $b = \frac{3 \,.\, 12 \,.\, 3 \,.\, 6}{8 \,.\, 22 \,.\, 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$; hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1{\cdot}7372}{1200} = \frac{1}{690}$ nearly. Consequently $v = 690$ is the velocity; and $t = \frac{1}{b}(\frac{1}{v} - \frac{1}{a}) = 2716 \times (\frac{1}{690} - \frac{1}{1200}) = 1\frac{31}{46}$ seconds, is the time required, or $1''$ and $\frac{2}{3}$ nearly.

PROBLEM XXII.

To determine the Relations of Space, Time, and Velocity, when a Globe descends, by its own Weight, in a Fluid.

THE foregoing notation remaining, viz. d = diameter, N and n the density of the ball and fluid, and v, s, t, the velocity, space, and time, in motion; we have $\frac{1}{6}pd^3$ = the magnitude of the ball, and $\frac{1}{6}pd^3(\text{N} - n)$ = its weight in the fluid, also $m = \frac{pnd^2v^2}{32g}$ = its resistance from the fluid; consequently $\frac{1}{6}pd^3(\text{N} - n) - \frac{pnd^2v^2}{32g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}\text{N}d^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{\text{N}} - \frac{3nv^2}{16g\text{N}d}$ for the accelerative force.

2. Hence $v\dot{v} = 2gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{2gf} = \frac{\text{N}v\dot{v}}{2g(\text{N} - n) - \frac{3n}{8d}v^2}$ $= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8\text{N}d}$, and $\frac{1}{a} = \frac{3n}{2g \,.\, 8d(\text{N} - n)}$, or $ab = 2g$ nearly; the fluent of which is $s =$ - - - - $\frac{1}{2b} \times$ log. of $\frac{a}{a - v^2}$, an expression for the space s, in terms of the velocity v. That is, when s and v begin, or are equal to nothing, both together.

But if the body commence motion in the fluid with a certain given velocity e, or enter the fluid with that velocity, like as when the body, after falling in empty space from a certain height, falls into a fluid like water; then the correct fluent will be $s = \frac{1}{2b} \times$ hyp. log. of $\frac{a - e^2}{a - v^2}$.

3. But now, to determine v in terms of s, put c = 2·718281828; then, since the log. of $\frac{a}{a - v^2} = 2bs$, therefore $\frac{a}{a - v^2} = c^{2bs}$, or $\frac{a - v^2}{a} = c^{-2bs}$; hence $v =$ - - - $\sqrt{a - ac^{-2bs}}$ is the velocity sought.

4. The

4. The greatest velocity is to be found, as in art. 5 of prob. 20, by making $\dot{f}$ or $1 - \frac{n}{N} - \frac{3nv^2}{16gNd} = 0$, which gives $v = \sqrt{(2g \cdot 8d \cdot \frac{N - n}{3n})} = \sqrt{a}$. The same value of v is obtained by making the fluxion of v^2, or of $a - ac^{-2bs}$, $= 0$. And the same value of v is also obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{a - ac^{-2bs}}$ continually approaches. It is evident however, that it will approximate towards it the faster, the greater b is, or the less d is; and that, the diameters being very small, the bodies descend by nearly uniform velocities, which are directly in the subduplicate ratio of the diameters. See also art. 5, prob. 20, for other observations on this head.

5. To find the time t. Now $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{1 - c^{-2bs}}}$. Then, to find the fluent of this fluxion, put $z = \sqrt{1 - c^{-2bs}} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1 - z^2}$, consequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1 - z^2}$, - - - - and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + z}{1 - z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a} + v}{\sqrt{a} - v}$, which is the general expression for the time.

EXAM. If it were required to determine the time and velocity, by descending in air 1000 feet, the ball being of lead, and 1 inch diameter.

Here $N = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

Hence $a = \frac{2 \cdot 16\frac{1}{12} \cdot \frac{8}{12} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} = \frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}$, and $b = \frac{3 \cdot \frac{3}{2500}}{8 \cdot 11\frac{1}{3} \cdot \frac{1}{12}} = \frac{3 \cdot 3 \cdot 3 \cdot 12}{8 \cdot 34 \cdot 2500} = \frac{9 \cdot 9}{68}$;

consequently $v = \sqrt{a} \times \sqrt{1 - c^{-2bs}} = \sqrt{\frac{193 \cdot 34 \cdot 50^2}{9 \cdot 27}} \times \sqrt{(1 - c^{-\frac{81}{85}})} = 203\frac{2}{3}$ the velocity. And $t = \frac{1}{2b\sqrt{a}} \times \log.$

2 +

$$\frac{2 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \sqrt{\frac{34 \,.\, 2500}{27 \,.\, 193}} \times \log. \frac{1\cdot78383}{0\cdot21617} = 8\cdot5236'',$$

the time.

Note. If the globe be so light as to ascend in the fluid; it is only necessary to change the signs of the first two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{16gNd}$; and then proceeding in all respects as before.

SCHOLIUM.

To compare this theory, contained in the last four problems, with experiment, the few following numbers are here extracted from extensive tables of velocities and resistances, resulting from a course of many hundred very accurate experiments, made in the course of the year 1786.

In the first column are contained the mean uniform or greatest velocities acquired in air, by globes, hemispheres, cylinders, and cones, all of the same diameter, and the altitude of the cone nearly equal to the diameter also, when urged by the several weights expressed in avoirdupois ounces, and standing on the same line with the velocities, each in their proper column. So, in the first line, the numbers show, that, when the greatest or uniform velocity was accurately 3 feet per second, the bodies were urged by these weights, according as their different ends went foremost; namely, by ·028 oz. when the vertex of the cone went foremost; by ·064 oz. when the base of the cone went foremost; by ·027 oz. for a whole sphere; by ·050 oz. for a cylinder; by ·051 oz. for the flat side of the hemisphere; and by ·020 oz. for the round or convex side of the hemisphere. Also, at the bottom of all, are placed the mean proportions of the resistances of these figures in the nearest whole numbers. Note, the common diameter of all the figures, was 6·375, or $6\frac{3}{8}$ inches; so that the area of the circle of that diameter is just 32 square inches, or $\frac{2}{9}$ of a square foot; and the altitude of the cone was $6\frac{1}{2}$ inches. Also, the diameter of the small hemisphere was $4\frac{3}{4}$ inches, and consequently the area of its base $17\frac{3}{4}$ square inches, or $\frac{1}{8}$ of a square foot nearly.

From the given dimensions of the cone, it appears, that the angle made by its side and axis, or direction of the path, is 26 degrees, very nearly.

The

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62°; consequently the weight of a cubic foot of air was equal to $1\frac{1}{5}$ oz. nearly, in those circumstances.

Veloc. per sec.	Cone.		Whole globe.	Cylinder.	Hemisphere.		Small Hemis. flat.
	vertex	base.			flat.	round.	
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.
3	·028	·064	·027	·050	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·297	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·816	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·057	2·528	2·542	1·033	1·196
Propor. Numb.	126	291	124	285	288	119	140

From this table of resistances, several practical inferences may be drawn. As,

1. That the resistance is nearly as the surface; the resistance increasing but a very little above that proportion in the greater surfaces. Thus, by comparing together the numbers in the 6th and last columns, for the bases of the two hemispheres, the areas of which are in the proportion of $17\frac{3}{4}$ to 32, or as 5 to 9 very nearly; it appears that the numbers in those two columns, expressing the resistances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the resistances on the greater surface increase gradually more and more above that proportion. And the mean resistances are as 140 to 288, or as 5 to

to $10\frac{2}{7}$. This circumstance therefore agrees nearly with the theory.

2. The resistance to the same surface, is nearly as the square of the velocity; but gradually increasing more and more above that proportion, as the velocity increases. This is manifest from all the columns. And therefore this circumstance also differs but little from the theory, in small velocities.

3. When the hinder parts of bodies are of different forms, the resistances are different, though the fore parts be alike; owing to the different pressures of the air on the hinder parts. Thus, the resistance to the fore part of the cylinder, is less than that on the flat base of the hemisphere, or of the cone; because the hinder part of the cylinder is more pressed or pushed, by the following air, than those of the other two figures.

4. The resistance on the base of the hemisphere, is to that on the convex side, nearly as $2\frac{2}{5}$ to 1, instead of 2 to 1, as the theory assigns the proportion. And the experimented resistance, in each of these, is nearly $\frac{1}{4}$ part more than that which is assigned by the theory.

5. The resistance on the base of the cone is to that on the vertex, nearly as $2\frac{3}{10}$ to 1. And in the same ratio is radius to the sine of the angle of the inclination of the side of the cone, to its path or axis. So that, in this instance, the resistance is directly as the sine of the angle of incidence, the transverse section being the same, instead of the square of the sine.

6. Hence we can find the altitude of a column of air, whose pressure shall be equal to the resistance of a body, moving through it with any velocity. Thus,

Let a = the area of the section of the body, similar to any of those in the table, perpendicular to the direction of motion;
r = the resistance to the velocity, in the table; and
x = the altitude sought, of a column of air, whose base is a, and its pressure r.

Then ax = the content of the column in feet,
and $1\frac{1}{5}ax$ or $\frac{6}{5}ax$ its weight in ounces; - - - - - - -
therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude sought in

feet,

feet, namely, $\frac{5}{6}$ of the quotient of the resistance of any body divided by its transverse section; which is a constant quantity for all similar bodies, however different in magnitude, since the resistance r is as the section a, as was found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in the foregoing table, except the small hemisphere: then, $x = \frac{5}{6} \times \frac{r}{a}$ becomes $x = \frac{15}{4}r$, where r is the resistance in the table, to the similar body.

If, for example, we take the convex side of the large hemisphere, whose resistance is ·634 oz. to a velocity of 16 feet per second, then $r = {\cdot}634$, and $x = \frac{15}{4} r = 2{\cdot}3775$ feet, is the altitude of the column of air whose pressure is equal to the resistance on a spherical surface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$, the altitude due to the velocity 16; which is near double the altitude that is equal to the pressure. And as the altitude is proportional to the square of the velocity, therefore, in small velocities, the resistance to any spherical surface, is equal to the pressure of a column of air on its great circle, whose altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

But if the cylinder be taken, whose resistance $r = 1{\cdot}526$: then $x = \frac{15}{4}r = 5{\cdot}72$; which exceeds the height, 4, due to the velocity in the ratio of 23 to 16 nearly. And the difference would be still greater, if the body were larger; and also if the velocity were more.

7. Also, if it be required to find with what velocity any flat surface must be moved, so as to suffer a resistance just equal to the whole pressure of the atmosphere:

The resistance on the whole circle whose area is $\frac{2}{9}$ of a foot, is ·051 oz. with the velocity of 3 feet per second; it is $\frac{1}{9}$ of ·051, or ·0056 oz. only, with a velocity of 1 foot. But $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$ oz. is the whole pressure of the atmosphere. Therefore, as $\sqrt{{\cdot}0056} : \sqrt{7556} :: 1 : 1162$ nearly, which is the velocity sought. Being almost equal to the velocity with which air rushes into a vacuum.

8. Hence may be inferred the great resistance suffered by military projectiles. For, in the table, it appears, that a globe of $6\frac{3}{8}$ inches diameter, which is equal to the size of an iron ball weighing 36lb, moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of $\frac{2}{3}$ of an ounce weight; and therefore, computing

ing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417lb, and that independent of the pressure of the atmosphere itself on the fore part of the ball, which would be 487lb more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would be more than 900lb to such a velocity.

9. Having said, in the last article, that the pressure of the atmosphere is taken entirely off the hinder part of the ball moving with a velocity of 1600 feet per second; which must happen when the ball moves faster than the particles of air can follow by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent air: let us therefore inquire what this velocity is. Now the velocity with which any fluid issues, depends on its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the barometer to be 30 inches, or $2\frac{1}{2}$ feet, the height of a uniform atmosphere, all of the same density as at the earth's surface, would be $2\frac{1}{2} \times 13{\cdot}6 \times 833\frac{1}{3}$ or 28333 feet; therefore $\sqrt{16} : \sqrt{28333} :: 32 : 8\sqrt{28333} = 1346$ feet, which is the velocity sought. And therefore, with a velocity of 1600 feet per second, or any velocity above 1346 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere on its fore part, as well as the resistance arising from the *vis inertia* of the particles of air struck by the ball.

10. On the whole, we find that the resistance of the air, as determined by the experiments, differs very widely, both in respect to its quantity on all figures, and in respect to the proportions of it on oblique surfaces, from the same as determined by the preceding theory; which is the same as that of Sir Isaac Newton, and most modern philosophers. Neither should we succeed better if we have recourse to the theory given by Professor Gravesande, or others, as similar differences and inconsistencies still occur.

We conclude therefore, that all the theories of the resistance of the air hitherto given, are very erroneous. And the preceding one is only laid down, till further experiments, on this important subject, shall enable us to deduce from them another, that shall be more consonant to the true phænomena of nature.

LOGARITHMS

LOGARITHMS

OF THE

NUMBERS

FROM

1 to 1000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1.681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

N. B. In the following table, in the last nine columns of each page, where the first or leading figures change from 9's to 0's, points or dots are now introduced instead of the 0's through the rest of the line, to catch the eye, and to indicate that from thence the corresponding natural number in the first column stands in the next lower line, and its annexed first two figures of the Logarithm in the second column.

N.	0	1	2	3	4	5	6	7	8	9
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
102	8600	9026	9451	9876	.300	.724	1147	1570	1993	2415
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616
104	7033	7451	7868	8284	8700	9116	9532	9947	.361	.775
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
107	9384	9789	.195	.600	1004	1408	1812	2216	2619	3021
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028
109	7426	7825	8223	8620	9017	9414	9811	.207	.602	.998
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
112	9218	9606	9993	.380	.766	1153	1538	1924	2309	2694
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	.320
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
117	8186	8557	8928	9298	9668	..38	.407	.776	1145	1514
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	9181	9543	9904	.266	.626	.987	1347	1707	2067	2426
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
123	9905	.258	.611	.963	1315	1667	2018	2370	2721	3071
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	.026
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	.253
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	.245
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	..12
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
138	9879	.194	.508	.822	1136	1450	1763	2076	2389	2702
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911
141	9219	9527	9835	.142	.449	.756	1063	1370	1676	1982
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
144	8362	8664	8965	9266	9567	9868	.168	.469	.769	1068
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802

N.	0	1	2	3	4	5	6	7	8	9
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689
151	8977	9264	9552	9839	.126	.413	.699	.986	1272	1558
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	..51
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382
158	8657	8932	9206	9481	9755	..29	.303	.577	.850	1124
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
162	9515	9783	..51	.319	.586	.853	1121	1388	1654	1921
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	.193
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
173	8046	8297	8548	8799	9049	9299	9550	9800	..50	.300
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	.176
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610
179	2853	3095	3338	3580	3822	4064	4306	4548	4790	5031
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
185	7172	7406	7641	7875	8110	8344	8580	8812	9046	9279
186	9513	9746	9980	.213	.446	.679	.912	1144	1377	1609
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927
188	4158	4389	4620	4850	5031	5311	5542	5772	6002	6232
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	8754	8982	9211	9439	9667	9895	.123	.351	.578	.806
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
199	8853	9071	9289	9507	9725	9943	.161	.378	.595	.813

N.	0	1	2	3	4	5	6	7	8	9
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417
204	9630	9843	. . 56	. 268	. 481	. 693	. 906	1118	1330	1542
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
213	8380	8583	8787	8991	9194	9398	9601	9805	. . . 8	. 211
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
218	8456	8656	8855	9054	9253	9451	9650	9849	. . 47	. 246
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	. . 54
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646
229	9835	. . 25	. 215	. 404	. 593	. 783	. 972	1161	1350	1539
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
234	9216	9401	9587	9772	9958	. 143	. 328	. 513	. 698	. 883
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	. . 30
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
245	9166	9343	9520	9698	9875	. . 51	. 228	. 405	. 582	. 759
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766

N.	0	1	2	3	4	5	6	7	8	9
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501
251	9674	9847	..20	.192	.365	.538	.711	.883	1056	1228
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
257	9933	.102	.271	.440	.609	.777	.946	1114	1283	1451
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	4973	5140	5807	5474	5641	5808	5974	6141	6308	6474
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
263	9956	.121	.286	.451	.616	.781	.975	1110	1275	1439
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591
269	9752	9914	..75	.236	.398	.559	.720	.881	1042	1203
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004
273	6163	6322	6481	6640	6800	6957	7116	7275	7433	7592
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
275	9333	9491	9648	9806	9964	.122	.279	.437	.594	.752
276	440909	1066	1224	1381	1538	1695	1852	2209	2166	2323
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	..95
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
288	9392	9543	9694	9845	9995	.146	.296	.447	.597	.748
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
295	9822	9969	.116	.263	.410	.557	.704	.851	.998	1145
296	471292	1488	1585	1732	1878	2025	2171	2318	2464	2610
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976

N.	0	1	2	3	4	5	6	7	8	9
300	477121	7266	7411	7555	7700	7844	7989	8133	8278	8422
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818
309	9958	..99	.239	.380	.520	.661	.801	.941	1081	1222
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550
316	9687	9824	9962	..99	.236	.374	.511	.648	.785	.922
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068
323	9203	9337	9471	9606	9740	9874	...9	.143	.277	.411
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697
331	9828	9959	..90	.221	.353	.484	.615	.745	.876	1007
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	..72
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351
340	1479	1607	1734	1862	1990	2117	2245	2372	2500	2627
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951
346	9076	9202	9327	9452	9578	9703	9829	9954	..79	.204
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701
		4950	3074	3199	3323	3447	3571	3696	3820	3944

N.	0	1	2	3	4	5	6	7	8	9
350	544068	4192	4316	4440	4564	4688	4812	4936	5060	5183
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	.106
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787
363	9907	..26	.146	.265	.385	.504	.624	.743	.863	.982
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257
371	9374	9491	9608	9725	9842	9959	..76	.193	.309	.426
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669
380	9784	9898	..12	.126	.241	.355	.469	.583	.697	.811
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838
389	9950	..61	.173	.284	.396	.507	.619	.730	.842	.953
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774
398	9883	9992	.101	.210	.319	.428	.537	.646	.755	.864
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951

N.	0	1	2	3	4	5	6	7	8	9
400	602060	2169	2277	2386	2494	2603	2711	2819	2928	3036
401	3144	3253	3361	3469	3573	3686	3794	3902	4010	4118
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488
407	9594	9701	9808	9914	..21	.128	.234	.341	.447	.554
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	..32
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308
426	9410	9512	9613	9715	9817	9919	..21	.123	.224	.326
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387
436	9486	9586	9686	9785	9885	9984	..84	.183	.283	.382
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237
446	9335	9432	9530	9627	9724	9821	9919	..16	.113	.210
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116

N.	0	1	2	3	4	5	6	7	8	9
450	653213	3309	3405	3502	3598	3695	3791	3888	3984	4080
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042
452	5138	5235	5331	5427	5526	5619	5715	5810	5906	6002
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821
457	9916	..11	.106	.201	.296	.391	.486	.581	.676	.771
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663
460	2758	2852	2947	3041	3135	3230	3324	3418	3512	3607
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224
467	9317	9410	9503	9596	9689	9782	9875	9967	..60	.153
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337
478	9428	9519	9610	9700	9791	9882	9973	..63	.154	.245
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151
480	1241	1332	1422	1513	603	1693	1784	1874	1964	2055
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220
489	9309	9398	9486	9575	9664	9753	9841	9930	..19	.107
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883

N.	0	1	2	3	4	5	6	7	8	9
500	698970	9057	9144	9231	9317	9404	9491	9578	9664	9751
501	9838	9924	..11	..98	.184	.271	.358	.444	.531	.617
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	..33
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566
516	2650	2734	2818	2902	2986	3070	3154	3238	3326	3407
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	..77
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893
537	9974	..55	.136	.217	.298	.378	.459	.540	.621	.702
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493
549	9572	9651	9731	9810	9889	9968	..47	.126	.205	.284

N	0	1	2	3	4	5	6	7	8	9
550	740363	0442	0521	0560	0678	0757	0836	0915	0994	1073
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659
562	9736	9814	9891	9968	..45	.123	.200	.277	.354	.431
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592
575	9668	9743	9819	9894	9970	..45	.121	.196	.272	.347
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353
580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	..42
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079

N.	0	1	2	3	4	5	6	7	8	9
600	778151	8224	8296	8368	8441	8513	8585	8658	8730	8802
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524
602	9596	9669	9741	9813	9885	9957	..29	.101	.173	.245
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510
616	9581	9651	9722	9792	9863	9933	...4	..74	.144	.215
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272
630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492
645	9560	9627	9694	9762	9829	9896	9964	..31	..98	.165
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847

N.	0	1	2	3	4	5	6	7	8	9
650	812913	2980	3047	3114	3181	3247	3314	3381	3448	3514
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478
660	9544	9610	9676	9741	9807	9873	9939	...4	..70	.136
661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409
666	3474	3539	3605	3670	3735	3800	3865	3930	3995	4061
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882
676	9947	..11	..75	.139	.204	.268	.332	.396	.460	.525
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261
686	6324	6387	6451	0514	6577	6641	6704	6767	6830	6894
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	..43
692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	50[illegible]

N.	0	1	2	3	4	5	6	7	8	9
700	845098	5160	5222	5284	5346	5408	5470	5532	5594	5656
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
704	7573	7634	7676	7758	7819	7881	7943	8004	8066	8128
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679
724	9739	9799	9859	9918	9978	..38	..98	.158	.218	.278
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
735	6287	6346	6405	5465	6524	6583	6642	6701	6760	6819
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760
741	9818	9877	9935	9994	..53	.111	.170	.228	.287	.345
742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003

N	0	1	2	3	4	5	6	7	8	9
750	875061	5119	5177	5235	5293	5351	5409	5466	5524	5582
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612
758	9669	9726	9784	9841	9898	9956	..13	..70	.127	.185
759	880242	0299	0356	0413	0471	0528	0585	0542	0699	0756
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172
766	4229	5285	4342	4399	4455	4512	4569	4625	4682	4739
767	4795	4852	4909	4965	5022	5078	5185	5192	5248	5305
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998
771	7054	7111	7167	7233	7280	7336	7392	7449	7505	7561
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806
776	9862	9918	9974	..30	..86	.141	.197	.253	.309	.365
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706
783	3762	3817	3873	3928	3954	4039	4094	4150	4205	4261
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766
794	9821	9875	9930	9985	..39	..94	.149	.203	.258	.312
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036

N.	0	1	2	3	4	5	6	7	8	9
800	903090	3144	319[illegible]	3253	3307	3361	3416	3470	3524	3578
801	3633	368[illegible]	3741	3795	3849	3904	3958	4012	4066	4120
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
803	4716	477[illegible]	4824	4878	493[illegible]	4986	5040	5094	5148	5202
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
806	6335	6389	644[illegible]	6497	6551	6604	6658	6712	6766	6820
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967
811	9021	9074	9128	9181	9235	9289	93[illegible]2	9396	9449	9503
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	..37
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169
817	2222	2275	2323	2381	2435	2488	2541	2594	2647	2700
818	2753	2806	2859	2913	2966	3019	30[illegible]2	3125	3178	3231
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819
822	4872	4925	497[illegible]	5030	5083	5136	5189	5241	5294	5347
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978
828	8030	7083	8185	8188	8240	8293	8345	8397	8450	8502
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549
831	9601	9653	9706	9758	9810	9862	9914	9967	..19	..71
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593
883	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114
884	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
887	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805
845	6857	6908	695[illegible]	70[illegible]1	7062	7114	7165	7216	7268	7319
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832
847	7883	7935	798[illegible]	8037	8088	8140	8191	8242	8293	8345
848	839[illegible]	8447	849[illegible]	8549	8601	8652	8703	8754	8805	8857
	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368

N.	0	1	2	3	4	5	6	7	8	9
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879
851	9930	9981	..32	..83	.134	.185	.236	.287	.338	.389
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5900
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
865	7016	7066	7117	7167	7217	7207	7317	7367	7418	7468
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470
868	8520	8570	8620	8670	8720	8779	8820	8870	8919	8970
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
873	1014	1064	1114	1163	1213	1203	1313	1362	1412	1402
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
877	3000	3049	3099	3148	3198	3247	3297	3316	3396	3445
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8365
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829
891	9878	9926	9975	..24	..73	.121	.170	.219	.267	.316
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
895	1823	1872	1920	1969	2017	2066	2114	2103	2211	2260
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194

N.	0	1	2	3	4	5	6	7	8	9
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
912	9995	..42	..90	.138	.185	.233	.280	.328	.376	.423
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
933	9882	9928	9975	..21	..68	.114	.161	.207	.254	.300
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678

N.	0	1	2	3	4	5	6	7	8	9
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
967	5426	5471	5516	5561	5606	5651	5699	5741	5786	5830
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
975	9005	9049	9049	9138	9183	9227	9272	9316	9361	9405
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
977	9895	9939	9983	..28	..72	.117	.161	.206	.250	.294
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
994	7886	7430	7474	7517	7561	7605	7648	7692	7736	7779
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957

	0 Deg.				1 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0		10·000000			8.241855	9.999934	8.241921	11·758079	60
1	6·463726	10·000000	6·463726	13·536274	8·249033	9·999932	8·249102	11·750898	59
2	6·764756	10·000000	6·764756	13·235244	8·256094	9·999929	8·256165	11·743835	58
3	6·940847	10·000000	6·940847	13·059153	8·263042	9·999927	8·263115	11·736885	57
4	7·065786	10·000000	7·065786	12·934214	8·269881	9·999925	8·269956	11·730044	56
5	7·162696	10·000000	7·162696	12·837304	8·276614	9·999922	8·276691	11·723309	55
6	7·241877	9·999999	7·241878	12·758122	8·283243	9·999920	8·283323	11·716677	54
7	7·308824	9·999999	7·308825	12·691175	8·289773	9·999918	8·289856	11·710144	53
8	7·366816	9·999999	7·366817	12·633183	8·296207	9·999915	8·296292	11·703708	52
9	7·417968	9·999999	7·417970	12·582030	8·302546	9·999913	8·302634	11·697366	51
10	7·463726	9·999998	7·463727	12·536273	8·308794	9·999910	8·308884	11·691116	50
11	7·505118	9·999998	7·505120	12·494880	8·314954	9·999907	8·315046	11·684954	49
12	7·542906	9·999997	7·542909	12·457091	8·321027	9·999905	8·321122	11·678878	48
13	7·577668	9·999997	7·577672	12·422328	8·327016	9·999902	8·327114	11·672886	47
14	7·609853	9·999996	7·609857	12·390143	8·332924	9·999899	8·333025	11·666975	46
15	7·639816	9·999996	7·639820	12·360180	8·338753	9·999897	8·338856	11·661144	45
16	7·667845	9·999995	7·667849	12·332151	8·344504	9·999894	8·344610	11·655390	44
17	7·694173	9·999995	7·694179	12·305821	8·350181	9·999891	8·350289	11·649711	43
18	7·718997	9·999994	7·719003	12·280997	8·355783	9·999888	8·355895	11·644105	42
19	7·742478	9·999993	7·742484	19·257516	8·361315	9·999885	8·361430	11·638570	41
20	7·764754	9·999993	7·764761	12·235239	8·366777	9·999882	8·366895	11·633105	40
21	7·785943	9·999992	7·785951	12·214049	8·372171	9·999879	8·372292	11·627708	39
22	7·806146	9·999991	7·806155	12·193845	8·377499	9·999876	8·377622	11·622378	38
23	7·825451	9·999990	7·825460	12·174540	8·382762	9·999873	8·382889	11·617111	37
24	7·843934	9·999989	7·843944	12·156056	8·387962	9·999870	8·388092	11·611908	36
25	7·861662	9·999989	7·861674	12·138326	8·393101	9·999867	8·393234	11·606766	35
26	7·878695	9·999988	7·870708	12·121292	8·398179	9·999864	8·398315	11·601685	34
27	7·895085	9·999987	7·895099	12·104901	8·403199	9·999861	8·403338	11·596662	33
28	7·910879	9·999986	7·910894	12·089106	8·408161	9·999858	8·408304	11·591696	32
29	7·926119	9·999985	7·926134	12·073866	8·413068	9·999854	8·413213	11·586787	31
30	7·940842	9·999983	7·940858	12·059142	8·417919	9·999851	8·418068	11·581932	30
31	7·955082	9·999982	7·955100	12·044900	8·422717	9·999848	8·422869	11·577131	29
32	7·968870	9·999981	7·968889	12·031111	8·427462	9·999845	8·427618	11·572382	28
33	7·982233	9·999980	7·982253	12·017747	8·432156	9·999841	8·432315	11·567685	27
34	7·995198	9·999979	7·995219	12·004881	8·436800	9·999838	8·436962	11·563038	26
35	8·007787	9·999977	8·007809	11·992191	8·441394	9·999834	8·441560	11·558440	25
36	8·020021	9·999976	8·020044	11·979956	8·445941	9·999831	8·446110	11·553890	24
37	8·031919	9·999975	8·031945	11·968055	8·450440	9·999827	8·450613	11·549387	23
38	8·043501	9·999973	8·043527	11·956473	8·454893	9·999824	8·455070	11·544930	22
39	8·054781	9·999972	8·054809	11·945191	8·459301	9·999820	8·459481	11·540519	21
40	8·065776	9·999971	8·065806	11·934194	8·463665	9·999816	8·463849	11·536151	20
41	8·076500	9·999969	8·076531	11·923469	8·467985	9·999813	8·468172	11·531828	19
42	8·086965	9·999968	8·086997	11·913003	8·472263	9·999809	8·472454	11·527546	18
43	8·097183	9·999966	8·097217	11·902783	8·476498	9·999805	8·476693	11·523307	17
44	8·107167	9·999964	8·107203	11·892797	8·480693	9·999801	8·480892	11·519108	16
45	8·116926	9·999963	8·116963	11·883037	8·484848	9·999797	8·485050	11·514950	15
46	8·126471	9·999961	8·126510	11·873490	8·488963	9·999794	8·489170	11·510830	14
47	8·135810	9·999959	8·135851	11·864149	8·493040	9·999790	8·493250	11·506750	13
48	8·144953	9·999958	8·144996	11·855004	8·497078	9·999786	8·497293	11·502707	12
49	8·153907	9·999956	8·153952	11·846048	8·501080	9·999782	8·501298	11·498702	11
50	8·162681	9·999954	8·162727	11·837273	8·505045	9·999778	8·505267	11·494733	10
51	8·171280	9·999952	8·171328	11·828672	8·508974	9·999774	8·509200	11·490800	9
52	8·179713	9·999950	8·179763	11·820237	8·512867	9·999769	8·513098	11·486902	8
53	8·187985	9·999948	8·188036	11·811964	8·516726	9·999765	8·516961	11·483039	7
54	8·196102	9.999946	8·196156	11·803844	8·520551	9·999761	8·520790	11·479210	6
55	8·204070	9·999944	8·204126	11·765874	8·524343	9·999757	8·524586	11·475414	5
56	8·211895	9·999942	8·211953	11·788047	8·528102	9·999753	8·528349	11·471651	4
57	8·219581	9·999940	8·219641	11·780359	8·531828	9·999748	8·532080	11·467920	3
58	8·227134	9·999938	8·227195	11·772805	8·535523	9·999744	8·535779	11·464221	2
59	8·234557	9·999936	8·234621	11·765379	8·539186	9·999740	8·539447	11·460553	1
60	8·241855	9·999934	8·241921	11·758079	8·542819	9·999735	8·543084	11·456916	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	
	89 Deg				88 Deg				

	2 Deg.				3 Deg.				
′	Sine.	Cosine.	Tang	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	8·54281	9·999735	8·54308	11·456916	8·718800	9·999404	8·719396	11·280604	60
1	8·546422	9·999731	8·546691	11·453309	8·721204	9·999398	8·721806	11·278194	59
2	8·549995	9·999726	8·550268	11·449732	8·723595	9 999391	8·724204	11·275796	58
3	8·55353	9·999722	553817	1·446183	8·725972	9·999384	8·726588	11·273412	57
4	8·557034	9·999717	8·55733[illegible]	11·442664	8 728337	9·999378	3·728959	11·271041	56
5	8·560540	9 999713	8·560828	11·439179	8·730688	9·999371	8·731317	11·268683	55
6	8·563999	9 999708	8·564291	1·435709	8·733027	9·999364	8·733663	11·266337	54
7	8·567431	9·999704	8·567727	11·432273	8·735354	9 999357	8·735996	11·264004	53
8	8·570836	9·999699	8·571137	11·428863	8·737667	9·999350	8·738317	11·261683	52
9	8 574214	9·999694	8 574520	11·425480	8·739969	9·999343	8·740626	11·259374	51
10	8·577566	9 999689	8·577877	11·422123	8·742259	9·999336	8·742922	11 257078	50
11	8·580892	9·999685	8·581208	11 418792	8·744536	9·999329	8·745207	11·254793	49
12	8·584193	9·999680	8 58 514	11·415486	8·746802	9·999322	8·747479	11·252521	48
13	8·587469	9 999675	8·587795	11·412205	8 749055	9·999315	8·749740	11·250260	47
14	8·590721	9·999670	8·591051	11·40 949	8 751297	9·999308	8·751989	11·248011	46
15	8·593948	9·999665	8 59428	11·405717	8·753528	9 999301	8·754227	11·245773	45
16	8·597152	9·999660	8·597492	11 402508	8·755747	9·999294	8·756453	11·243547	44
17	8·600332	9·999655	8·600677	11·399323	8·757955	9·999287	8·758668	11·241332	43
18	8· 03489	9·999650	8·603839	11·396161	8·760151	9·999279	8·760872	11·239128	42
19	8·606623	9·999645	8·606978	11·393022	8·762337	9·999272	8·763065	11·236935	41
20	8·609734	9·999640	8·610094	11·389909	8·764511	9·999265	8·765246	11·234754	40
21	8·612823	9·999635	8·613189	11·386811	8·766675	9·999257	8·767417	11·232583	39
22	8·615891	9·999629	8·616262	11·383738	8·768828	9·999250	8·769578	11·230422	38
23	8 618937	9·999624	8·619313	11·380687	8·770970	9·999242	8·771727	11·228273	37
24	621962	9·999619	8·622343	11·377657	8·773101	9·999235	8·773866	11·226134	36
25	8·624965	9·999614	8·625352	11·374648	8·775223	9·999227	8·775995	11·224005	35
26	8·627948	9 999608	8·628340	11·371660	8·777333	9·999220	8·778114	11·221886	34
27	8·630911	9·999603	8·631308	11·368692	8·779434	9·999212	8·780222	11·219778	33
28	8 633854	9·999597	8·634256	11·365744	8·781524	9 999205	8·782320	11·217680	32
29	8·636776	9·999592	8.637184	11·562816	8·783605	9·999197	8·784408	11·215592	31
30	8·639680	9·999586	640093	11·359907	8·785675	9·999189	8·786486	11·213514	30
31	8·642563	9·999581	8·642982	11·357018	8·787736	9·999181	8 788554	11·211446	29
32	8·645 28	9·999575	8·645853	11·354147	8·789787	9·999174	8·790613	11·209337	28
33	8·648274	9·999570	8·648704	11·351296	8·791828	9·999166	8·792662	11·207338	27
34	8·651102	9·999564	8·651537	11·348463	8·793859	9·999158	8·794701	11·205299	26
35	8·653911	9·999558	8·654352	11·345648	8·795881	9·999150	8·796731	11·203269	25
36	8·656702	9·999553	8·657149	11·342851	8·797894	9·999142	8·798752	11·201248	24
37	8·659475	·999547	8·659928	11·340072	8·799897	9·999134	8·800763	11·199237	23
38	8·662230	9 999541	8·662689	11·337311	8·801892	9·999126	8·802765	11·197235	22
39	8·664968	6·999535	8·665433	11·334567	8·803876	9·999118	8·804758	11·195242	21
40	8·667689	9·999529	8·668160	11·331840	8·805852	9·999110	8·806742	11·193258	20
41	8·670393	·999524	8·670870	11·329130	8·807819	9·999102	8·808717	11·191283	19
42	8·673080	9·999518	8·673563	11·326437	8·809777	9·999094	8·810683	11·189317	18
43	8·675751	9·999512	8 676239	11·323761	8·811726	9·999086	8·812641	11·187359	17
44	8·678405	9·999506	8·678900	11·321100	8·813667	9·999077	8·814589	11·185411	16
45	8·681043	9·999500	8·681544	11·318456	8·815599	9·999069	8·816529	11·183471	15
46	8·68 665	9·999493	8·684172	11·315828	8 817522	9·999061	8·818461	11·181539	14
47	8·686272	9·999487	8·686784	11·313216	8·819436	9·999053	8·820384	11·179616	13
48	8 688863	9·999481	8·689381	11·310619	8·821343	9·999044	8·822298	11·177702	12
49	8·691438	9·999475	8·691963	11·308037	8·823230	9·999036	8·824205	11·175795	11
50	8·693998	9·999469	8·694529	11·305471	8 825130	9·999027	8·826103	11·173897	10
51	8·696543	9·999463	8 697081	11·302919	8 827011	9·999019	8·827992	11·172008	9
52	8·699073	9·999456	8·699617	11 300383	8·828884	9·999010	8·829874	11·170126	8
53	8 01589	9 999450	8·702139	11·297861	8·830749	9·999002	8·831748	11·168252	7
54	8·704090	9·999443	8·704646	11·295354	8·832607	9·998993	8·833613	11·166387	6
55	8·706577	9·999437	8·707140	11·292860	8·834456	9·998984	8·835471	11·164529	5
56	8·709049	9·999431	8 709618	11·290382	8·836297	9·998976	8·837321	11·162679	4
57	8·711507	9·999424	8·712083	11·287917	8·838130	9·998967	8·839163	11·160837	3
58	8·713952	9·999418	8·714534	11·285466	8·839956	9·998958	8·840998	11·159002	2
59	8·716383	9·999411	8·716972	11·283028	8·841774	9·998950	8·842825	11·157175	1
60	8·718800	9·999404	8·719396	11·280604	8·843585	9·998941	8·844644	11·155356	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	T	
	87 Deg.				86 Deg.				

	4 Deg.				5 Deg.				
'	Sine.	Cosine.	Tang.	Cotang.	Sine.	Cosine.	Tang.	Cotang.	
0	8·843585	9·998941	8·844644	11·155356	8·940296	9·998344	8·941952	11·058048	60
1	8·845387	9·998932	8·846455	11·153545	8·941738	9·998333	8·943404	11·056596	59
2	8·847183	9·998923	8·848260	11·151740	8·943174	9·998322	8·944852	11·055148	58
3	8·848971	9·998914	8·850057	11·149943	8·944606	9·998311	8·946295	11·053705	57
4	8·850751	9·998905	8·851846	11·148154	8·946034	9·998300	8·947734	11·052266	56
5	8·852525	9·998896	8·853628	11·146372	8·947456	9·998289	8·949168	11·050832	55
6	8·854291	9·998887	8·855403	11·144597	8·948874	9·998277	8·950597	11·049403	54
7	8·856049	9·998878	8·857171	11·142829	8·950287	9·998266	8·952021	11·047979	53
8	8·857801	9·998869	8·858932	11·141068	8·951696	9·998255	8·953441	11·046559	52
9	8·859546	9·998860	8·860686	11·139314	8·953100	9·998243	8·954856	11·045144	51
10	8·861283	9·998851	8·862433	11·137567	8·954499	9·998232	8·956267	11·043733	50
11	8·863014	9·998841	8·864173	11·135827	8·955894	9·998220	8·957674	11·042326	49
12	8·864738	9·998832	8·865906	11·134094	8·957284	9·998209	8·959075	11·040925	48
13	8·866455	9·998823	8·867632	11·132368	8·958670	9·998197	8·960473	11·039527	47
14	8·868165	9·998813	8·869351	11·130649	8·960052	9·998186	8·961866	11·038134	46
15	8·869868	9·998804	8·871064	11·128936	8·961429	9·998174	8·963255	11·036745	45
16	8·871565	9·998795	8·872770	11·127230	8·962801	9·998163	8·964639	11·035361	44
17	8·873255	9·998785	8·874469	11·125531	8·964170	9·998151	8·966019	11·033981	43
18	8·874938	9·998776	8·876162	11·123838	8·965534	9·998139	8·967394	11·032606	42
19	8·876615	9·998766	8·877849	11·122151	8·966895	9·998128	8·968766	11·031234	41
20	8·878285	9·998757	8·879529	11·120471	8·968249	9·998116	8·970133	11·029867	40
21	8·879949	9·998747	8·881202	11·118798	8·969600	9·998104	8·971496	11·028504	39
22	8·881607	9·998738	8·882869	11·117131	8·970947	9·998092	8·972855	11·027145	38
23	8·883258	9·998728	8·884530	11·115470	8·972289	9·998080	8·974209	11·025791	37
24	8·884903	9·998718	8·886185	11·113815	8·973628	9·998068	8·975560	11·024440	36
25	8·886542	9·998708	8·887833	11·112167	8·974962	9·998056	8·976906	11·023094	35
26	8·888174	9·998699	8·889476	11·110524	8·976293	9·998044	8·978248	11·021752	34
27	8·889801	9·998689	8·891112	11·108888	8·977619	9·998032	8·979586	11·020414	33
28	8·891421	9·998679	8·892742	11·107258	8·978941	9·998020	8·980921	11·019079	32
29	8·893035	9·998669	8·894366	11·105634	8·980259	9·998008	8·982251	11·017749	31
30	8·894643	9·998659	8·895984	11·104016	8·981573	9·997996	8·983577	11·016423	30
31	8·896246	9·998649	8·897596	11·102404	8·982883	9·997984	8·984899	11·015101	29
32	8·897842	9·998639	8·899203	11·100797	8·984189	9.997972	8·986217	11·013783	28
33	8·899432	9·998629	8·900803	11·099197	8·985491	9·997959	8·987532	11·012468	27
34	8·901017	9·998619	8·902398	11·097602	8·986789	9·997947	8·988842	11·011158	26
35	8·902596	9·998609	8·903987	11·096013	8·988083	9·997935	8·990149	11·009851	25
36	8·904169	9·998599	8·905570	11·094430	8·989374	9·997922	8·991451	11·008549	24
37	8·905736	9·998589	8·907147	11·092853	8·990660	9·997910	8·992750	11·007250	23
38	8·907297	9·998578	8·908719	11·091281	8·991943	9·997897	8·994045	11·005955	22
39	8·908853	9·998568	8·910285	11·089715	8·993222	9·997885	8·995337	11·004663	21
40	8·910404	9·998558	8·911846	11·088154	8·994497	9·997872	8·996624	11·003376	20
41	8·911949	9·998548	8·913401	11·086599	8·995768	9·997860	8·997908	11·002092	19
42	8·913488	9·998537	8·914951	11·085049	8·997036	9·997847	8·999188	11·000812	18
43	8·915022	9·998527	8·916495	11·083505	8·998299	9·997835	9·000465	10·999535	17
44	8·916550	9·998516	8·918034	11·081966	8·999560	9·997822	9·001738	10·998262	16
45	8·918073	9·998506	8·919568	11·080432	9·000816	9·997809	9·003007	10·996993	15
46	8·919591	9·998495	8·921096	11·078904	9·002069	9·997797	9·004272	10·995728	14
47	8·921103	9·998485	8·922619	11·077381	9·003318	9·997784	9·005534	10·994466	13
48	8·922610	9·998474	8·924136	11·075864	9·004563	9·997771	9·006792	10·993208	12
49	8·924112	9·998464	8·925649	11·074351	9·005805	9·997758	9·008047	10·991953	11
50	8·925609	9·998453	8·927156	11·072844	9·007044	9·997745	9·009298	10·990702	10
51	8·927100	9·998442	8·928658	11·071342	9·008278	9·997732	9·010546	10·989454	9
52	8·928587	9·998431	8·930155	11·069845	9·009510	9·997719	9·011790	10·988210	8
53	8·930068	9·998421	8·931647	11·068353	9·010737	9·997706	9·013031	10·986969	7
54	8·931544	9·998410	8·933134	11·066866	9·011962	9·997693	9·014268	10·985732	6
55	8·933015	9·998399	8·934616	11·065384	9·013182	9·997680	9·015502	10·984498	5
56	8·934481	9·998388	8·936093	11·063907	9·014400	9·997667	9·016732	10·983268	4
57	8·935942	9·998377	8·937565	11·062435	9·015613	9·997654	9·017959	10·982041	3
58	8·937398	9·998366	8·939032	11·060968	9·016824	9·997641	9·019183	10·980817	2
59	8·938850	9·998355	8·940494	11·059506	9·018031	9·997628	9·020403	10·979597	1
	8·940296	9·998344	8·941952	11·058048	9·019235	9·997614	9·021620	10·978380	0
	Cosine.	Sine.	Cotan.	Tang.	Cosine.	Sine.	Cotan.	Tang.	'
	85 Deg.				84 Deg.				

	6 Deg.				7 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·019235	9·997614	9·021620	10·978380	9·085894	9.996751	9·089144	10·910856	60
1	9·020435	9·997601	9·022834	10·977166	9·086922	9·996735	9·090187	10·909813	59
2	9·021632	9·997588	9·024044	10·975956	9·087947	9·996720	9·091228	10·908772	58
3	9·022825	9·997574	9·025251	10·974749	9·088970	9·996704	9·092262	10·907734	57
4	9·024016	9·997561	9·026455	10·973545	9·089990	9·996688	9·093302	10·906698	56
5	9·025203	9·997547	9·027655	10·972345	9·091006	9·996673	9·094336	10·905664	55
6	9·026386	9·997534	9·028852	10·971148	9·092024	9·996657	9·095367	10·904633	54
7	9·027567	9·997520	9·030046	10·969954	9·093037	9·996641	9·096395	10·903605	53
8	9·028744	9·997507	9·031237	10·968763	9·094047	9·996625	9·097422	10·902578	52
9	9·029918	9·997493	9·032425	10·967575	9·095056	9·996610	9·098448	10·901554	51
10	9·031089	9·997480	9·033609	10·966391	9·096062	9·996594	9·099468	10·900532	50
11	9·032257	9·997466	9·034791	10·965209	9·097065	9·996578	9·100487	10·899513	49
12	9·033421	9·997452	9·035969	10·964031	9·098066	9·996562	9·101504	10·898496	48
13	9·034582	9·997439	9·037144	10·962856	9·099065	9·996546	9·102519	10·897481	47
14	9·035741	9·997425	9·038316	10·961684	9·100062	9·996530	9·103532	10·896468	46
15	9·036896	9·997411	9·039485	10·960515	9·101056	9·996514	9·104542	10·895458	45
16	9·038048	9·997397	9·040651	10·959349	9·102048	9·996498	9·105550	10·894450	44
17	9·039197	9·997383	9·041813	10·958187	9·103037	9·996482	9·106556	10·893444	43
18	9·040342	9·997369	9·042973	10·957027	9·104025	9·996465	9·107559	10·892441	42
19	9·041485	9·997355	9·044130	10·955870	9·105010	9·996449	9·108560	10·891440	41
20	9·042625	9·997341	9·045284	10·954716	9·105992	9·996433	9·109559	10·890441	40
21	9·043762	9·997327	9·046434	10·953566	9·106973	9·996417	9·110556	10·889444	39
22	9·044895	9·997313	9·047582	10·952418	9·107951	9·996400	9·111551	10·8884[illegible]	38
23	9·046026	9·997299	9·048727	10·951273	9·108927	9·996384	9·112543	10·887457	37
24	9·047154	9·997285	9·049869	10·950131	9·109901	9·996368	9·113533	10·886467	36
25	9·048277	9·997271	9·051008	10·948992	9·110873	9·966351	9·114521	10·885479	35
26	9·049400	9·997257	9·052144	10·947856	9·111842	9·996335	9·115507	10·884493	34
27	9·050519	9·997242	9·053277	10·946723	9·112809	9·996318	9·116491	10·883509	33
28	9·051635	9·997228	9·054407	10·945593	9·113774	9·996302	9·117472	10·882528	32
29	9·052749	9·997214	9·055535	10·944465	9·114737	9·996285	9·118452	10·881548	31
30	9·053859	9·997199	9·056659	10·943341	9·115698	9·996269	9·119429	10·880571	30
31	9·054966	9·997185	9·057781	10·942219	9·116656	9·996252	9·120404	10·879596	29
32	9·056071	9·997170	9·058900	10·941100	9·117613	9·996235	9·121377	10·878623	28
33	9·057172	9·997156	9·060016	10·939984	9·118567	9·996219	9·122348	10·877652	27
34	9·058271	9·997141	9·061130	10·938870	9·119519	9·996202	9·123317	10·876683	26
35	9·059367	9·997127	9·062240	10·937760	9·120469	9·996185	9·124284	10·875716	25
36	9·060460	9·997112	9·063348	10·936652	9·121417	9·996168	9·125249	10·874751	24
37	9·061551	9·997098	9·064453	10·935547	9·122362	9·996151	9·126211	10·873789	23
38	9·062639	9·997083	9·065556	10·934444	9·123306	9·996134	9·127172	10·872828	22
39	9·063724	9·997068	9·066655	10·933345	9·124248	9·996117	9·128130	10·871870	21
40	9·064806	9·997053	9·067752	10·932248	9·125187	9·996100	9·129087	10·870913	20
41	9·065885	9·997039	9·068846	10·931154	9·126125	9·996083	9·130041	10·869959	19
42	9·066962	9·997024	9·069938	10·930062	9·127060	9·996066	9·130994	10·869006	18
43	9·068036	9·997009	9·071027	10·928973	9·127993	9·996049	9·131944	10·868056	17
44	9·069107	9·996994	9·072113	10·927887	9·128925	9·996032	9·132893	10·867107	16
45	9·070176	9·996979	9·073197	10·926803	9·129854	9·996015	9·133839	10·866161	15
46	9·071242	9·996964	9·074278	10·925722	9·130781	9·995998	9·134784	10·865216	14
47	9·072306	9·996949	9·075356	10·924644	9·131706	9·995980	9·135726	10·864274	13
48	9·073366	9·996934	9·076432	10·923568	9·132630	9·995963	9·136667	10·863333	12
49	9·074424	9·996919	9·077505	10·922495	9·133551	9·995946	9·137605	10·862395	11
50	9·075480	9·996904	9·078576	10·921424	9·134470	9·995928	9·138542	10·861458	10
51	9·076533	9·996889	9·079644	10·920356	9·135387	9·995911	9·139476	10·860524	9
52	9·077583	9·996874	9·080710	10·919290	9·136303	9·995894	9·140409	10·859591	8
53	9·078631	9·996858	9·081775	10·918227	9·137216	9·995876	9·141340	10·858660	7
54	9·079676	9·996843	9·082833	10·917167	9·138128	9·995859	9·142269	10·857731	6
55	9·080719	9·996828	9·083891	10·916109	9·139037	9·995841	9·143196	10·856804	5
56	9·081759	9·996812	9·084947	10·915053	9·139944	9·995823	9·144121	10·855879	4
57	9·082797	9·996797	9·086000	10·914000	9·140850	9·995806	9·145044	10·854956	3
58	9·083832	9·996782	9·087050	10·912950	9·141754	9·995788	9·145966	10·854034	2
59	9·084864	9·996766	9·088098	10·911902	9·142655	9·995771	9·146885	10·853115	1
60	9·085894	9·996751	9·089144	10·910856	9·143555	9·995753	9·147803	10·852197	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang	
	83 Deg.				82 Deg.				

	8 Deg.				9 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	′
0	9·143555	9·995753	9·147803	10·852197	9·194332	9·994620	9·199713	10·800287	60
1	9·144453	9·995735	9·148718	10·851282	9·195129	9·994600	9·200529	10·799471	59
2	9·145349	9·995717	9·149632	10·850368	9·195925	9·994580	9·201345	10·798655	58
3	9·146243	9·995699	9·150544	10·849456	9·196719	9·994560	9·202159	10·797841	57
4	9·147136	9·995681	9·151454	10·848546	9·197511	9·994540	9·202971	10·797029	56
5	9·148026	9·995664	9·152363	10·847637	9·198302	9·994519	9·203782	10·796218	55
6	9·148915	9·995646	9·153269	10·846731	9·199091	9·994499	9·204592	10·795408	54
7	9·149802	9·995628	9·154174	10·845826	9·199879	9·994479	9·205400	10·794600	53
8	9·150686	9·995610	9·155077	10·844923	9·200666	9·994459	9·206207	10·793793	52
9	9·151569	9·995591	9·155978	10·844022	9·201451	9·994438	9·207013	10·792987	51
10	9·152451	9·995573	9·156877	10·843123	9·202234	9·994418	9·207817	10·792183	50
11	9·153330	9·995555	9·157775	10·842225	9·203017	9·994398	9·208619	10·791381	49
12	9·154208	9·995537	9·158671	10·841329	9·203797	9·994377	9·209420	10·790580	48
13	9·155083	9·995519	9·159565	10·840435	9·204577	9·994357	9·210220	10·789780	47
14	9·155957	9·995501	9·160457	10·839543	9·205354	9·994336	9·211018	10·788982	46
15	9·156830	9·995482	9·161347	10·838653	9·206131	9·994316	9·211815	10·788185	45
16	9·157700	9·995464	9·162236	10·837764	9·206906	9·994295	9·212611	10·787389	44
17	9·158569	9·995446	9·163123	10·836877	9·207679	9·994274	9·213405	10·786595	43
18	9·159435	9·995427	9·164008	10·835992	9·208452	9·994254	9·214198	10·785802	42
19	9·160301	9·995409	9·164892	10·835108	9·209222	9·994233	9·214989	10·785011	41
20	9·161164	9·995390	9·165774	10·834226	9·209992	9·994212	9·215780	10·784220	40
21	9·162025	9·995372	9·166654	10·833346	9·210760	9·994191	9·216568	10·783432	39
22	9·162885	9·995353	9·167532	10·832468	9·211526	9·994171	9·217356	10·782644	38
23	9·163743	9·995334	9·168409	10·831591	9·212291	9·994150	9·218142	10·781858	37
24	9·164600	9·995316	9·169284	10·830716	9·213055	9·994129	9·218926	10·781074	36
25	9·165454	9·995297	9·170157	10·829843	9·213818	9·994108	9·219710	10·780290	35
26	9·166307	9·995278	9·171029	10·828971	9·214579	9·994087	9·220492	10·779508	34
27	9·167159	9·995260	9·171899	10·828101	9·215338	9·994066	9·221272	10·778728	33
28	9·168008	9·995241	9·172767	10·827233	9·216097	9·994045	9·222052	10·777948	32
29	9·168856	9·995222	9·173634	10·826366	9·216854	9·994024	9·222830	10·777170	31
30	9·169702	9·995203	9·174499	10·825501	9·217609	9·994003	9·223607	10·776393	30
31	9·170547	9·995184	9·175362	10·824638	9·218363	9·993982	9·224382	10·775618	29
32	9·171389	9·995165	9·176224	10·823776	9·219116	9·993960	9·225156	10·774844	28
33	9·172230	9·995146	9·177084	10·822916	9·219868	9·993939	9·225929	10·774071	27
34	9·173070	9·995127	9·177942	10·822058	9·220618	9·993918	9·226700	10·773300	26
35	9·173908	9·995108	9·178799	10·821201	9·221367	9·993897	9·227471	10·772529	25
36	9·174744	9·995089	9·179655	10·820345	9·222115	9·993875	9·228239	10·771761	24
37	9·175578	9·995070	9·180508	10·819492	9·222861	9·993854	9·229007	10·770993	23
38	9·176411	9·995051	9·181360	10·818640	9·223606	9·993832	9·229773	10·770227	22
39	9·177242	9·995032	9·182211	10·817789	9·224349	9·993811	9·230539	10·769461	21
40	9·178072	9·995013	9·183059	10·816941	9·225092	9·993789	9·231302	10·768698	20
41	9·178900	9·994993	9·183907	10·816093	9·225833	9·993768	9·232065	10·767935	19
42	9·179726	9·994974	9·184752	10·815248	9·226573	9·993746	9·232826	10·767174	18
43	9·180551	9·994955	9·185597	10·814403	9·227311	9·993725	9·233586	10·766414	17
44	9·181374	9·994935	9·186439	10·813561	9·228048	9·993703	9·234345	10·765655	16
45	9·182196	9·994916	9·187280	10·812720	9·228784	9·993681	9·235103	10·764897	15
46	9·183016	9·994896	9·188120	10·811880	9·229518	9·993660	9·235859	10·764141	14
47	9·183834	9·994877	9·188958	10·811042	9·230252	9·993638	9·236614	10·763386	13
48	9·184651	9·994857	9·189794	10·810206	9·230984	9·993616	9·237368	10·762632	12
49	9·185466	9·994838	9·190629	10·809371	9·231715	9·993594	9·238120	10·761880	11
50	9·186280	9·994818	9·191462	10·808538	9·232444	9·993572	9·238872	10·761128	10
51	9·187092	9·994798	9·192294	10·807706	9·233172	9·993550	9·239622	10·760378	9
52	9·187903	9·994779	9·193124	10·806876	9·233899	9·993528	9·240371	10·759629	8
53	9·188712	9·994759	9·193953	10·806047	9·234625	9·993506	9·241118	10·758882	7
54	9·189519	9·994739	9·194780	10·805220	9·235349	9·993484	9·241865	10·758135	6
55	9·190325	9·994720	9·195606	10·804394	9·236073	9·993462	9·242610	10·757390	5
56	9·191130	9·994700	9·196430	10·803570	9·236795	9·993440	9·243354	10·756646	4
57	9·191933	9·994680	9·197253	10·802747	9·237515	9·993418	9·244097	10·755903	3
58	9·192734	9·994660	9·198074	10·801926	9·238235	9·993396	9·244839	10·755161	2
59	9·193534	9·994640	9·198894	10·801106	9·238953	9·993374	9·245579	10·754421	1
60	9·194332	9·994620	9·199713	10·800287	9·239670	9·993351	9·246319	10·753681	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	81 Deg.				80 Deg.				

	10 Deg.				11 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·239670	9·993351	9·246319	10·753681	9·280599	9·991947	9·288652	10·711348	60
1	9·240386	9·993329	9·247057	10·752943	9·281248	9·991922	9·289326	10·710674	59
2	9·241101	9·993307	9·247794	10·752206	9·281897	9·991897	9·289999	10·710001	58
3	9·241814	9·993284	9·248530	10·751470	9·282544	9·991873	9·290671	10·709329	57
4	9·242526	9·993262	9·249264	10·750736	9·283190	9·991848	9·291342	10·708658	56
5	9·243237	9·993240	9·249998	10·750002	9·283836	9·991823	9·292013	10·707987	55
6	9·243947	9·993217	9·250730	10·749270	9·284480	9·991799	9·292682	10·707318	54
7	9·244656	9·993195	9·251461	10·748539	9·285124	9·991774	9·293350	10·706650	53
8	9·245363	9·993172	9·252191	10·747809	9·285766	9·991749	9·294017	10·705983	52
9	9·246069	9·993149	9·252920	10·747080	9·286408	9·991724	9·294684	10·705316	51
10	9·946775	9·993127	9·253648	10·746352	9·287048	9·991699	9·295349	10·704651	50
11	9·247478	9·993104	9·254374	10·745626	9·287688	9·991674	9·296013	10·703987	49
12	9·248181	9·993081	9·255100	10·744900	9·288326	9·991649	9·296677	10·703323	48
13	9·248883	9·993059	9·255824	10·744176	9·288964	9·991624	9·297339	10·702661	47
14	9·249583	9·993036	9·256547	10·743453	9·289600	9·991599	9·298001	10·701999	46
15	9·250282	9·993013	9·257269	10·742731	9·290236	9·991574	9·298662	10·701338	45
16	9·250980	9·992990	9·257990	10·742010	9·290870	9·991549	9·299322	10·700678	44
17	9·251677	9·992967	9·258710	10·741290	9·291504	9·991524	9·299980	10·700020	43
18	9·252373	9·992944	9·259429	10·740571	9·292137	9·991498	9·300638	10·699362	42
19	9·253067	9·992921	9·260146	10·739854	9·292768	9·991473	9·301295	10·698705	41
20	9·253761	9·992898	9·260863	10·739137	9·293399	9·991448	9·301951	10·698049	40
21	9·254453	9·992875	9·261578	10·738422	9·294029	9·991422	9·302607	10·697393	39
22	9·255144	9·992852	9·262292	10·737708	9·294658	9·991397	9·303261	10·696739	38
23	9·255834	9·992829	9·263005	10·736995	9·295286	9·991372	9·303914	10·696086	37
24	9·256523	9·992806	9·263717	10·736283	9·295913	9·991346	9·304567	10·695433	36
25	9·257211	9·992783	9·264428	10·735572	9·296539	9·991321	9·305218	10·694782	35
26	9·257898	9·992759	9·265138	10·734862	9·297164	9·991295	9·305869	10·694131	34
27	9·258583	9·992736	9·265847	10·734153	9·297788	9·991270	9·306519	10·693481	33
28	9·259268	9·992713	9·266555	10·733445	9·298412	9·991244	9·307168	10·692832	32
29	9·259951	9·992690	9·267261	10·732739	9·299034	9·991218	9·307816	10·692184	31
30	9·260633	9·992666	9·267967	10·732033	9·299655	9·991195	9·308463	10·691537	30
31	9·261314	9·992643	9·268671	10·731329	9·300276	9·991167	9·309109	10·690891	29
32	9·261994	9·992619	9·269375	10·730625	9·300895	9·991141	9·309754	10·690246	28
33	9·262673	9·992596	9·270077	10·729925	9·301514	9·991115	9·300399	10·689601	27
34	9·263351	9·992572	9·270779	10·729221	9·302132	9·991090	9·311042	10·688958	26
35	9·264027	9·992549	9·271479	10·728521	9·302748	9·991064	9·311685	10·688315	25
36	9·264703	9·992525	9·272178	10·727822	9·303364	9·991038	9·312327	10·687673	24
37	9·265377	9·992501	9·272876	10·727124	9·303979	9·991012	9·312968	10·687032	23
38	9·266051	9·992478	9·273573	10·726427	9·304593	9·990986	9·313608	10·686392	22
39	9·266723	9·992454	9·274269	10·725731	9·305207	9·990960	9·314247	10·685753	21
40	9·267395	9·992430	9·274964	10·725036	9·305819	9·990934	9·314885	10·685115	20
41	9·268065	9·992406	9·275658	10·724342	9·306430	9·990908	9·315523	10·684477	19
42	9·268734	9·992382	9·276351	10·723649	9·307041	9·990882	9·316159	10·683841	18
43	9·269402	9·992359	9·277043	10·722957	9·307650	9·990855	9·316795	10·683205	17
44	9·270069	9·992335	9·277734	10·722266	9·308259	9·990829	9·317430	10·682570	16
45	9·270735	9·992311	9·278424	10·721576	9·308867	9·990803	9·318064	10·681936	15
46	9·271400	9·992287	9·279113	10·720887	9·309474	9·990777	9·318697	10·681303	14
47	9·272064	9·992263	9·279801	10·720199	9·310080	9·990750	9·319330	10·680670	13
48	9·272726	9·992239	9·280488	10·719512	9·310685	9·990724	9·319961	10·680039	12
49	9·273388	9·992214	9·281174	10·718826	9·311289	9·990697	9·320592	10·679408	11
50	9·274049	9·992190	9·281858	10·718142	9·311893	9·990671	9·321222	10·678778	10
51	9·274708	9·992166	9·282542	10·717458	9·312495	9·990645	9·321851	10·678149	9
52	9·275367	9·992142	9·283225	10·716775	9·313097	9·990618	9·322479	10·677521	8
53	9·276025	9·992118	9·283907	10·716093	9·313698	9·990591	9·323106	10·676894	7
54	9·276681	9·992093	9·284588	10·715412	9·314297	9·990565	9·323733	10·676267	6
55	9·277337	9·992069	9·285268	10·714732	9·314897	9·990538	9·324358	10·675642	5
56	9·277991	9·992044	9·285947	10·714053	9·315495	9·990511	9·324983	10·675017	4
57	9·278645	9·992020	9·286624	10·713376	9·316092	9·990485	9·325607	10·674393	3
58	9·279297	9·991996	9·287301	10·712699	9·316689	9·990458	9·326231	10·673769	2
59	9·279948	9·991971	9·287977	10·712023	9·317284	9·990431	9·326853	10·673147	1
60	9·280599	9·991947	9·288652	10·711348	9·317879	9·990404	9·327475	10·672525	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	79 Deg.				78 Deg.				

	12 Deg.				13 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·317879	9·990404	9·327475	10·672525	9·352088	9·988724	9·363364	10·636636	60
1	9·318473	9·990378	9·328095	10·671905	9·352635	9·988695	9·363940	10·636060	59
2	9·319066	9·990351	9·328715	10·671285	9·353181	9·988666	9·364515	10·635485	58
3	9·319658	9·990324	9·329334	10·670666	9·353726	9·988636	9·365090	10·634910	57
4	9·320249	9·990297	9·329953	10·670047	9·354271	9·988607	9·365664	10·634336	56
5	9·320840	9·990270	9·330570	10·669430	9·354815	9·988578	9·366237	10·633763	55
6	9·321430	9·990243	9·331187	10·668813	9·355358	9·988548	9·366810	10·633190	54
7	9·322019	9·990215	9·331803	10·668197	9·355901	9·988519	9·367382	10·632618	53
8	9·322607	9·990188	9·332418	10·667582	9·356443	9·988489	9·367953	10·632047	52
9	9·323194	9·990161	9·333033	10·666967	9·356984	9·988460	9·368524	10·631476	51
10	9·323780	9·990134	9·333646	10·666354	9·357524	9·988430	9·369094	10·630906	50
11	9·324366	9·990107	9·334259	10·665741	9·358064	9·988401	9·369663	10·630337	49
12	9·324950	9·990079	9·334871	10·665129	9·358603	9·988371	9·370232	10·629768	48
13	9·325534	9·990052	9·335482	10·664518	9·359141	9·988342	9·370799	10·629201	47
14	9·326117	9·990025	9·336093	10·663907	9·359678	9·988312	9·371367	10·628633	46
15	9·326700	9·989997	9·336702	10·663298	9·360215	9·988282	9·371933	10·628067	45
16	9·327281	9·989970	9·337311	10·662689	9·360752	9·988252	9·372499	10·627501	44
17	9·327862	9·989942	9·337919	10·662081	9·361287	9·988223	9·373064	10·626936	43
18	9·328442	9·989915	9·338527	10·661473	9·361822	9·988193	9·373629	10·626371	42
19	9·329021	9·989887	9·339133	10·660867	9·362356	9·988163	9·374193	10·625807	41
20	9·329599	9·989860	9·339739	10·660261	9·362889	9·988133	9·374756	10·625244	40
21	9·330176	9·989832	9·340344	10·659656	9·363422	9·988103	9·375319	10·624681	39
22	9·330753	9·989804	9·340948	10·659052	9·363954	9·988073	9·375881	10·624119	38
23	9·331329	9·989777	9·341552	10·658448	9·364485	9·988043	9·376442	10·623558	37
24	9·331905	9·989749	9·342155	10·657845	9·365016	9·988013	9·377003	10·622997	36
25	9·332478	9·989721	9·342757	10·657243	9·365546	9·987983	9·377563	10·622437	35
26	9·333051	9·989693	9·343358	10·656642	9·366075	9·987953	9·378122	10·621878	34
27	9·333624	9·989665	9·343958	10·656042	9·366604	9·987922	9·378681	10·621319	33
28	9·334195	9·989637	9·344558	10·655442	9·367131	9·987892	9·379239	10·620761	32
29	9·334767	9·989610	9·345157	10·654843	9·367659	9·987862	9·379797	10·620203	31
30	9·335337	9·989582	9·345755	10·654245	9·368185	9·987832	9·380354	10·619646	30
31	9·335906	9·989553	9·346353	10·653647	9·368711	9·987801	9·380910	10·619090	29
32	9·336475	9·989525	9·346949	10·653051	9·369236	9·987771	9·381466	10·618534	28
33	9·337043	9·989497	9·347545	10·652455	9·369761	9·987740	9·382020	10·617980	27
34	9·337610	9·989469	9·348141	10·651859	9·370285	9·987710	9·382575	10·617425	26
35	9·338176	9·989441	9·348735	10·651265	9·370808	9·987679	9·383129	10·616871	25
36	9·338742	9·989413	9·349329	10·650671	9·371330	9·987649	9·383682	10·616318	24
37	9·339307	9·989385	9·349922	10·650078	9·371852	9·987618	9·384234	10·615766	23
38	9·339871	9·989356	9·350514	10·649486	9·372373	9·987588	9·384786	10·615214	22
39	9·340434	9·989328	9·351106	10·648894	9·372894	9·987557	9·385337	10·614663	21
40	9·340996	9·989300	9·351697	10·648303	9·373414	9·987526	9·385888	10·614112	20
41	9·341558	9·989271	9·352287	10·647713	9·373933	9·987496	9·386438	10·613562	19
42	9·342119	9·989243	9·352876	10·647124	9·374452	9·987465	9·386987	10·613013	18
43	9·342679	9·989214	9·353465	10·646535	9·374970	9·987434	9·387536	10·612464	17
44	9·343239	9·989186	9·354053	10·645947	9·375487	9·987403	9·388084	10·611916	16
45	9·343797	9·989157	9·354640	10·645360	9·376003	9·987372	9·388631	10·611369	15
46	9·344355	9·989128	9·355227	10·644773	9·376519	9·987341	9·389178	10·610822	14
47	9·344912	9·989100	9·355813	10·644187	9·377035	9·987310	9·389724	10·610276	13
48	9·345469	9·989071	9·356398	10·643602	9·377549	9·987279	9·390270	10·609730	12
49	9·346024	9·989042	9·356982	10·643018	9·378063	9·987248	9·390815	10·609185	11
50	9·346579	9·989014	9·357566	10·642434	9·378577	9·987217	9·391360	10·608640	10
51	9·347134	9·988985	9·358149	10·641851	9·379089	9·987186	9·391903	10·608097	9
52	9·347687	9·988956	9·358731	10·641269	9·379601	9·987155	9·392447	10·607553	8
53	9·348240	9·988927	9·359313	10·640687	9·380113	9·987124	9·392989	10·607011	7
54	9·348792	9·988898	9·359893	10·640107	9·380624	9·987092	9·393531	10·606469	6
55	9·349343	9·988869	9·360474	10·639526	9·381134	9·987061	9·394073	10·605927	5
56	9·349893	9·988840	9·361053	10·638947	9·381643	9·987030	9·394614	10·605386	4
57	9·350443	9·988811	9·361632	10·638368	9·382152	9·986998	9·395154	10·604846	3
58	9·350992	9·988782	9·362210	10·637790	9·382661	9·986967	9·395694	10·604306	2
59	9·351540	9·988753	9·362787	10·637213	9·383168	9·986936	9·396233	10·603767	1
60	9·352088	9·988724	9·363364	10·636636	9·383675	9·986904	9·396771	10·603229	0
	Cosine	Sine	Cotan.	Tang	Cosine	Sine	Cotan.	Tang.	′
	77 Deg.				76 Deg.				

	14 Deg.				15 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·383675	9.986904	9·396771	10·603229	9·412996	9·984944	9·428052	10·571948	60
1	9·384182	9·986873	9·397309	10·602691	9·413467	9·984910	9·428558	10·571442	59
2	9·384687	9·986841	9·397846	10 602154	9 413938	9·984876	9·429062	10·570938	58
3	9.385192	9·986809	9·398383	10·601617	9·414408	9·984842	9 429566	10·570434	57
4	9·385697	9·986778	9·398919	10·601081	9 414878	9·984808	9·430070	10·569930	56
5	9·386201	9·986746	9·399455	10·600545	9·415347	9·984774	9·430573	10·569427	55
6	9·386704	9·986714	9·399990	10·600010	9 415815	9·984740	9·431075	10·568925	54
7	9 387207	9·986683	9·400524	10·599476	9·416283	9·984706	9·431577	10·568423	53
8	9·387709	9·986651	9·401058	10 598942	9·416751	9·984672	9 432079	10·567921	52
9	9·388210	9·986619	9·401591	10·598409	9·417217	9·984638	9·432580	10·567420	51
10	9·388711	9·986587	9·402124	10·597876	9·417684	9·984603	9·433080	10·566920	50
11	9·389211	9.986555	9·402656	10·597344	9·418150	9·984569	9·433580	10 566420	49
12	9·389711	9·986523	9 403187	10·596813	9·418615	9·984535	9·434080	10·565920	48
13	9·390210	9·986491	9·403718	10·596282	9·419079	9 984500	9·434579	10·565421	47
14	9 390705	9·986459	9·404249	10·595751	9·419544	9·984466	9·435078	10·564922	46
15	9·391206	9·986427	9·404778	10·595222	9·420007	9·984432	9·435576	10·564424	45
16	9·391703	9 986395	9·405308	10·594692	9·420470	9·984397	9·436073	10·563927	44
17	9·392199	9·986363	9·405836	10·594164	9·420933	9·984363	9·436570	10·563430	43
18	9·392695	9 986331	9·406364	10·593636	9·421395	9·984328	9·437067	10·562933	42
19	9·593191	9·986299	9·406892	10·593108	9·421857	9·984294	9·437563	10·562437	41
20	9·393685	9·986266	9·407419	10·592581	9·422318	9·984259	9·438059	10·561941	40
21	9·394179	9·986234	9·407945	10 592055	9·422778	9·984224	9·438554	10·561446	39
22	9 394673	9·986202	9·408471	10·591529	9·423238	9·984190	9·439048	10·560952	38
23	9·395166	9·986169	9·408996	10·591004	9·423697	9·984155	9·439563	10·560457	37
24	9·395658	9·986137	9·409521	10·590479	9·424156	9·984120	9·440036	10·559964	36
25	9·396150	9·986104	9·410045	10·589955	9·424615	9·984085	9·440529	10·559471	35
26	9·396641	9·986072	9·410569	10·589431	9·425073	9·984050	9·441022	10·558978	34
27	9·397132	9·986039	9·411092	10·588908	9·425530	9·984015	9·441514	10·558486	33
28	9·397621	9·986007	9·411615	10·588385	9·425987	9·983981	9·442006	10·557994	32
29	9·398111	9·985974	9·412137	10·587863	9·426443	9·983946	9·442497	10·557503	31
30	9·398600	9·985942	9·412658	10·587342	9·426899	9·983911	9·442988	10·557012	30
31	9·399088	9·985909	9·413179	10·586821	9·427354	9·983875	9·443479	10·556521	29
32	9·399575	9·985876	9·413699	10·586301	9·427809	9·983840	9·443968	10·556032	28
33	9·400062	9·985843	9·414219	10·585781	9·428263	9·983805	9·444458	10·555542	27
34	9·400549	9·985811	9·414738	10·585262	9·428717	9·983770	9·444947	10·555053	26
35	9·401035	9 985778	9·415257	10·584743	9·429170	9·983735	9·445435	10·554565	25
36	9·401520	9·985745	9·415775	10·584225	9·429623	9·983700	9·445923	10·554077	24
37	9·402005	9·985712	9·416293	10·583707	9·430075	9·983664	9·446411	10·553589	23
38	9·402489	9·985679	9·416810	10·583190	9·430527	9·983629	9·446898	10·553102	22
39	9·402972	9 985646	9·417326	10·582674	9·430978	9·983594	9·447384	10·552616	21
40	9·403455	9·985613	9·417842	10·582158	9·431429	9·983558	9·447870	10·552130	20
41	9 403938	9·985580	9·418358	10·581642	9·431879	9·983523	9·448356	10·551644	19
42	9·404420	9·985547	9·418873	10·581127	9·432329	9·983487	9·448841	10·551159	18
43	9·404901	9·985514	9·419387	10·580613	9·432778	9·983452	9·449326	10·550674	17
44	9·405382	9·985480	9·419901	10·580099	9·433226	9·983416	9·449810	10·550190	16
45	9·405862	9·985447	9·420415	10·579585	9·433675	9·983381	9·450294	10·549706	15
46	9·406341	9·985414	9·420927	10·579073	9·434122	9·983345	9·450777	10·549223	14
47	9·406820	9·985381	9·421440	10·578560	9·434569	9·983309	9·451260	10·548740	13
48	9·407299	9·985347	9·421952	10·578048	9·435016	9·983273	9·451743	10·548257	12
49	9·407777	9·985314	9·422463	10·577537	9·435462	9·983238	9·452225	10·547775	11
50	9·408254	9·985280	9·422974	10·577026	9·435908	9·983202	9·452706	10·547294	10
51	2·408731	9·985247	9·423484	10·576516	9 436353	9·983166	9·453187	10 546813	9
52	9·409207	9·985213	9·423993	10·576007	9·436798	9·983130	9·453668	10·546332	8
53	9·409682	9·985180	9·424503	10·575497	9·437242	9·983094	9·454148	10·545852	7
54	9·410157	9 985146	9·425011	10·574989	9·437686	9·983058	9·454628	10·545372	6
55	9·410632	9·985113	9·425519	10·574481	9·438129	9·983022	9·455107	10·544893	5
56	9·411106	9·985079	9 426027	10·573973	9·438572	9·982986	9·455586	10·544414	4
57	9·411579	9·985045	9·426534	10·573469	9·439014	9·982950	9·456064	10·543936	3
58	9·412052	9·985011	9·427041	10·572959	9·439456	9·982914	9·456542	10·543458	2
59	9·412524	9·984978	9·427547	10·572453	9·439897	9·982878	9·457019	10·542981	1
60	9·412996	9·984944	9·428052	10·571948	9 440338	9·982842	9·457496	10·542504	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	
	75 Deg.				74 Deg.				

	16 Deg.				17 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang	Cotang.	
0	9·440338	9·982842	9·457496	10·542504	9·465935	9·980596	9·485339	10·514661	60
1	9·440778	9·982805	9·457973	10·542027	9·466348	9·980558	9·485791	10·514209	59
2	9·441218	9·982769	9·458449	10·541551	9·466761	9·980519	9·486242	10·513758	58
3	9·441658	9·982733	9·458925	10·541075	9·467173	9·980480	9·486693	10·513307	57
4	9·442096	9·982696	9·459400	10·540600	9·467585	9·980442	9·487143	10·512857	56
5	9·442535	9·982660	9·459875	10·540125	9·467996	9·980403	9·487593	10·512407	55
6	9·442973	9·982624	9·460349	10·539651	9·468407	9·980364	9·488043	10·511957	54
7	9·443410	9·982587	9·460823	10·539177	9·468817	9·980325	9·488492	10·511508	53
8	9·443847	9·982551	9·461297	10·538703	9·469227	9·980286	9·488941	10·511059	52
9	9·444284	9·982514	9·461770	10·538230	9·469637	9·980247	9·489390	10·510610	51
10	9·444720	9·982477	9·462242	10·537758	9·470046	9·980208	9·489838	10·510162	50
11	9·445155	9·982441	9·462715	10·537285	9·470455	9·980169	9·490286	10·509714	49
12	9·445590	9·982404	9·463186	10·536814	9·470863	9·980130	9·490733	10·509267	48
13	9·446025	9·982367	9·463658	10·536342	9·471271	9·980091	9·491180	10·508820	47
14	9·446459	9·982331	9·464128	10·535872	9·471679	9·980052	9·491627	10·508373	46
15	9·446893	9·982294	9·464599	10·535401	9·472086	9·980012	9·492073	10·507927	45
16	9·447326	9·982257	9·465069	10·534931	9·472492	9·979973	9·492519	10·507481	44
17	9·447759	9·982220	9·465539	10·534461	9·472898	9·979934	9·492965	10·507035	43
18	9·448191	9·982183	9·466008	10·533992	9·473304	9·979895	9·493410	10·506590	42
19	9·448623	9·982146	9·466477	10·533523	9·473710	9·979855	9·493854	10·506146	41
20	9·449054	9·982109	9·466945	10·533055	9·474115	9·979816	9·494299	10·505701	40
21	9·449485	9·982072	9·467413	10·532587	9·474519	9·979776	9·494743	10·505257	39
22	9·449915	9·982035	9·467880	10·532120	9·474923	9·979737	9·495186	10·504814	38
23	9·450345	9·981998	9·468347	10·531653	9·475327	9·979697	9·495630	10·504370	37
24	9·450775	9·981961	9·468814	10·531186	9·475730	9·979658	9·496073	10·503927	36
25	9·451204	9·981924	9·469280	10·530720	9·476133	9·979618	9·496515	10·503485	35
26	9·451632	9·981886	9·469746	10·530254	9·476536	9·979579	9·496957	10·503043	34
27	9·452060	9·981849	9·470211	10·529789	9·476938	9·979539	9·497399	10·502601	33
28	9·452488	9·981812	9·470676	10·529324	9·477340	9·979499	9·497841	10·502159	32
29	9·452915	9·981774	9·471141	10·528859	9·477741	9·979459	9·498282	10·501718	31
30	9·453342	9·981737	9·471605	10·528395	9·478142	9·979420	9·498722	10·501278	30
31	9·453768	9·981700	9·472069	10·527931	9·478542	9·979380	9·499163	10·500837	29
32	9·454194	9·981662	9·472532	10·527468	9·478942	9·979340	9·499603	10·500397	28
33	9·454619	9·981626	9·472995	10·527005	9·479342	9·979300	9·500042	10·499958	27
34	9·455044	9·981587	9·473457	10·526543	9·479741	9·979260	9·500481	10·499519	26
35	9·455469	9·981549	9·473919	10·526081	9·480140	9·979220	9·500920	10·499080	25
36	9·455893	9·981512	9·474381	10·525619	9·480539	9·979180	9·501359	10·498641	24
37	9·456316	9·981474	9·474842	10·525158	9·480937	9·979140	9·501797	10·498203	23
38	9·456739	9·981436	9·475303	10·524697	9·481334	9·979100	9·502235	10·497765	22
39	9·457162	9·981399	9·475763	10·524237	9·481731	9·979059	9·502672	10·497328	21
40	9·457584	9·981361	9·476223	10·523777	9·482128	9·979019	9·503109	10·496891	20
41	9·458006	9·981323	9·476683	10·523317	9·482525	9·978979	9·503546	10·496454	19
42	9·458427	9·981285	9·477142	10·522858	9·482921	9·978939	9·503982	10·496018	18
43	9·458848	9·981247	9·477601	10·522399	9·483316	9·978898	9·504418	10·495582	17
44	9·459268	9·981209	9·478059	10·521941	9·483712	9·978858	9·504854	10·495146	16
45	9·459688	9·981171	9·478517	10·521483	9·484107	9·978817	9·505289	10·494711	15
46	9·460108	9·981133	9·478975	10·521025	9·484501	9·978777	9·505724	10·494276	14
47	9·460527	9·981095	9·479432	10·520568	9·484895	9·978737	9·506159	10·493841	13
48	9·460946	9·981057	9·479889	10·520111	9·485289	9·978696	9·506593	10·493407	12
49	9·461364	9·981019	9·480345	10·519655	9·485682	9·978655	9·507027	10·492973	11
50	9·461782	9·980981	9·480801	10·519199	9·486075	9·978615	9·507460	10·492540	10
51	9·462199	9·980942	9·481257	10·518743	9·486467	9·978574	9·507893	10·492107	9
52	9·462616	9·980904	9·481712	10·518288	9·486860	9·978533	9·508326	10·491674	8
53	9·463032	9·980866	9·482167	10·517833	9·487251	9·978493	9·508759	10·491241	7
54	9·463448	9·980827	9·482621	10·517379	9·487643	9·978452	9·509191	10·490809	6
55	9·463864	9·980789	9·483075	10·516925	9·488034	9·978411	9·509622	10·490378	5
56	9·464279	9·980750	9·483529	10·516471	9·488424	9·978370	9·510054	10·489946	4
57	9·464694	9·980712	9·483982	10·516018	9·488814	9·978329	9·510485	10·489515	3
58	9·465108	9·980673	9·484435	10·515565	9·489204	9·978288	9·510916	10·489084	2
59	9·465522	9·980685	9·484887	10·515113	9·489593	9·978247	9·511346	10·488654	1
60	9·465935	9·980596	9·485339	10·514661	9·489982	9·978206	9·511776	10·488224	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	73 Deg.				72 Deg.				

	18 Deg.				19 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·489982	9·978206	9·511776	10·488224	9·512642	9·975670	9·536972	10·463028	60
1	9·490371	9·978165	9·512206	10·487794	9·513009	9·975627	9·537382	10·462618	59
2	9·490759	9·978124	9·512635	10·487365	9·513375	9·975583	9·537792	10·462208	58
3	9·491147	9·978083	9·513064	10·486936	9·513741	9·975539	9·538202	10·461798	57
4	9·491535	9·978042	9·513493	10·486507	9·514107	9·975496	9·538611	10·461389	56
5	9·491922	9·978001	9·513921	10·486079	9·514472	9·975452	9·539020	10·460980	55
6	9·492308	9·977959	9·614349	10·485651	9·514837	9·975408	9·539429	10·460571	54
7	9·492695	9·977918	9·514777	10·485223	9·515202	9·975365	9·539837	10·460163	53
8	9·493081	9·977877	9·515204	10·484796	9·515566	9·975321	9·540245	10·459755	52
9	9·493466	9·977835	9·515631	10·484369	9·515930	9·975277	9·540653	10·459347	51
10	9·493851	9·977794	9·516057	10·483943	9·516294	9·975233	9·541061	10·458939	50
11	9·494236	9·977752	9·516484	10·483516	9·516657	9·975189	9·541468	10·458532	49
12	9·494621	9·977711	9·516910	10·483090	9·517020	9·975140	9·541875	10·458125	48
13	9·495005	9·977669	9·517335	10·482665	9·517382	9·975101	9·542281	10·457719	47
14	9·495388	9·977628	9·517761	10·482239	9·517745	9·975057	9·542688	10·457312	46
15	9·495772	9·977586	9·518186	10·481814	9·518107	9·975013	9·543094	10·456906	45
16	9·496154	9·977544	9·518610	10·481390	9·518468	9·974969	9·543499	10·456501	44
17	9·496537	9·977503	9·519034	10·480966	9·518829	9·974925	9·543905	10·456095	43
18	9·496919	9·977461	9·519458	10·480542	9·519190	9·974880	9·544310	10·455690	42
19	9·497301	9·977419	9·519882	10·480118	9·519551	9·974836	9·544715	10·455285	41
20	9·497682	9·977377	9·520305	10·479695	9·519911	9·974792	9·545119	10·454881	40
21	9·498064	9·977335	9·520728	10·479272	9·520271	9·974748	9·545524	10·454476	39
22	9·498444	9·977293	9·521151	10·478849	9·520631	9·974708	9·545928	10·454072	38
23	9·498825	9·977251	9·521573	10·478427	9·520990	9·974659	9·546331	10·453669	37
24	9·499204	9·977209	9·521995	10·478005	9·521349	9·974614	9·546735	10·453265	36
25	9·499584	9·977167	9·522417	10·477583	9·521707	9·974570	9·547138	10·452862	35
26	9·499963	9·977125	9·522838	10·477162	9·522066	9·974525	9·547540	10·452460	34
27	9·500342	9·977083	9·523259	10·476741	9·522424	9·974481	9·547943	10·452057	33
28	9·500721	9·977041	9·523680	10·476320	9·522781	9·974436	9·548345	10·451655	32
29	9·501099	9·976999	9·524100	10·475900	9·523138	9·974391	9·548747	10·451253	31
30	9·501476	9·976957	9·524520	10·475480	9·523495	9·974347	9·549149	10·450851	30
31	9·501854	9·976914	9·524940	10·475060	9·523852	9·974302	9·549550	10·450450	29
32	9·502231	9·976872	9·525360	10·474641	9·524208	9·974257	9·549951	10·450049	28
33	9·502607	9·976830	9·525778	10·474222	9·524564	9·974212	9·550352	10·449648	27
34	9·502984	9·976787	9·526197	10·473803	9·524920	9·974167	9·550752	10·449248	26
35	9·503360	9·976745	9·526615	10·473385	9·525275	9·974122	9·551153	10·448847	25
36	9·503735	9·976702	9·527033	10·472967	9·525630	9·974077	9·551552	10·448448	24
37	9·504110	9·976660	9·527451	10·472549	9·525984	9·974032	9·551952	10·448048	23
38	9·504485	9·976617	9·527868	10·472132	9·526339	9·973987	9·552351	10·447649	22
39	9·504860	9·976574	9·528285	10·471715	9·526693	9·973942	9·552750	10·447250	21
40	9·505234	9·976532	9·528702	10·471298	9·527046	9·973897	9·553149	10·446851	20
41	9·505608	9·976489	9·529119	10·470881	9·527400	9·973852	9·553548	10·446452	19
42	9·505981	9·976446	9·529535	10·470465	9·527753	9·973807	9·555946	10·446054	18
43	9·506354	9·976404	9·529951	10·470049	9·528105	9·973761	9·554344	10·445656	17
44	9·506727	9·976361	9·530366	10·469634	9·528458	9·973716	9·554741	10·445259	16
45	9·507099	9·976318	9·530781	10·469219	9·528810	9·973671	9·555139	10·444861	15
46	9·507471	9·976275	9·531196	10·468804	9·529161	9·973625	9·555536	10·444464	14
47	9·507843	9·976232	9·531611	10·468389	9·529513	9·973580	9·555933	10·444067	13
48	9·508214	9·976189	9·532025	10·467975	9·529864	9·973535	9·556329	10·443671	12
49	9·508585	9·976146	9·532439	10·467561	9·530215	9·973489	9·556725	10·443275	11
50	9·508956	9·976103	9·532853	10·467147	9·530565	9·973444	9·557121	10·442879	10
51	9·509326	9·976060	9·533266	10·466734	9·530915	9·973398	9·557517	10·442483	9
52	9·509696	9·976017	9·533679	10·466321	9·531265	9·973352	9·557913	10·442087	8
53	9·510065	9·975974	9·534092	10·465908	9·531614	9·973307	9·558308	10·441692	7
54	9·510434	9·975930	9·534504	10·465496	9·531963	9·973261	9·558703	10·441297	6
55	9·510803	9·975887	9·534916	10·465084	9·532312	9·973215	9·559097	10·440903	5
56	9·511172	9·975844	9·535328	10·464672	9·532661	9·973169	9·559491	10·440509	4
57	9·511540	9·975800	9·535739	10·464261	9·533009	9·973124	9·559885	10·440115	3
58	9·511907	9·975757	9·536150	10·463850	9·533357	9·973078	9·560279	10·439721	2
59	9·512275	9·975714	9·536561	10·463439	9·533704	9·973032	9·560673	10·439327	1
60	9·512642	9·975670	9·536972	10·463028	9·534052	9·972986	9·561066	10·438934	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	71 Deg.				70 Deg.				

	20 Deg.				21 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·534052	9·972986	9·561066	10·438934	9·554329	9·970152	9·584177	10·415823	60
1	9·534399	9·972940	9·561459	10·438541	9·554658	9·970103	9·584555	10·415445	59
2	9·534745	9·972894	9·561851	10·438149	9·554987	9·970055	9·584932	10·415068	58
3	9·535092	9·972848	9·562244	10·437756	9·555315	9·970006	9·585309	10·414691	57
4	9·535438	9·972802	9·562636	10·437364	9·555643	9·969957	9·585686	10·414314	56
5	9·535783	9·972755	9·563028	10·436972	9·555971	9·969909	9·586062	10·413938	55
6	9·536129	9·972709	9·563419	10·436581	9·556299	9·969860	9·586439	10·413561	54
7	9·536474	9·972663	9·563811	10·436189	9·556626	9·969811	9·586815	10·413185	53
8	9·536818	9·972617	9·564202	10·435798	9·556953	9·969762	9·587190	10·412810	52
9	9·537163	9·972570	9·564593	10·435407	9·557280	9·969714	9·587566	10·412434	51
10	9·537507	9·972524	9·564983	10·435017	9·557606	9·969665	9·587941	10·412059	50
11	9·537851	9·972478	9·565373	10·434627	9·557932	9·969616	9·588316	10·411684	49
12	9·538194	9·972431	9·565763	10·434237	9·558258	9·969567	9·588691	10·411309	48
13	9·538538	9·972385	9·566153	10·433847	9·558583	9·969518	9·589066	10·410934	47
14	9·538880	9·972338	9·566542	10·433458	9·558909	9·969469	9·589440	10·410560	46
15	9·539223	9·972291	9·566932	10·433068	9·559234	9·969420	9·589814	10·410186	45
16	9·539565	9·972245	9·567320	10·432680	9·559558	9·969370	9·590188	10·409812	44
17	9·539907	9·972198	9·567709	10·432291	9·559883	9·969321	9·590562	10·409438	43
18	9·540249	9·972151	9·568098	10·431902	9·560207	9·969272	9·590935	10·409065	42
19	9·540590	9·972105	9·568486	10·431514	9·560531	9·969223	9·591308	10·408692	41
20	9·540931	9·972058	9·568873	10·431127	9·560855	9·969173	9·591681	10·408319	40
21	9·541272	9·972011	9·569261	10·430739	9·561178	9·969124	9·592054	10·407946	39
22	9·541613	9·971964	9·569648	10·430352	9·561501	9·969075	9·592426	10·407574	38
23	9·541953	9·971917	9·570035	10·429965	9·561824	9·969025	9·592799	10·407201	37
24	9·542293	9·971870	9·570422	10·429578	9·562146	9·968976	9·593171	10·406829	36
25	9·542632	9·971823	9·570809	10·429191	9·562468	9·968926	9·593542	10·406458	35
26	9·542971	9·971776	9·571195	10·428805	9·562790	9·968877	9·593914	10·406086	34
27	9·543310	9·971729	9·571581	10·428419	9·563112	9·968827	9·594285	10·405715	33
28	9·543649	9·971682	9·571967	10·428033	9·563433	9·968777	9·594656	10·405344	32
29	9·543987	9·971635	9·572352	10·427648	9·563755	9·968728	9·595027	10·404973	31
30	9·544325	9·971588	9·572738	10·427262	9·564075	9·968678	9·595398	10·404602	30
31	9·544663	9·971540	9·573123	10·426877	9·564396	9·968628	9·595768	10·404232	29
32	9·545000	9·971493	9·573507	10·426493	9·564716	9·968578	9·596138	10·403862	28
33	9·545338	9·971446	9·573892	10·426108	9·565036	9·968528	9·596508	10·403492	27
34	9·545674	9·971398	9·574276	10·425724	9·565356	9·968479	9·596878	10·403122	26
35	9·546011	9·971351	9·574660	10·425340	9·565676	9·968429	9·597247	10·402753	25
36	9·546347	9·971303	9·575044	10·424956	9·565995	9·968379	9·597616	10·402384	24
37	9·546683	9·971256	9·575427	10·424573	9·566314	9·968329	9·597985	10·402015	23
38	9·547019	9·971208	9·575810	10·424190	9·566632	9·968278	9·598354	10·401646	22
39	9·547354	9·971161	9·576193	10·423807	9·566951	9·968228	9·598722	10·401278	21
40	9·547689	9·971113	9·576576	10·423424	9·567269	9·968178	9·599091	10·400909	20
41	9·548024	9·971066	9·576959	10·423041	9·567587	9·968128	9·599459	10·400541	19
42	9·548359	9·971018	9·577341	10·422659	9·567904	9·968078	9·599827	10·400173	18
43	9·548693	9·970970	9·577723	10·422277	9·568222	9·968027	9·600194	10·399806	17
44	9·549027	9·970922	9·578104	10·421896	9·568539	9·967977	9·600562	10·399438	16
45	9·549360	9·970874	9·578486	10·421514	9·568856	9·967927	9·600929	10·399071	15
46	9·549693	9·970827	9·578867	10·421133	9·569172	9·967876	9·601296	10·398704	14
47	9·550026	9·970779	9·579248	10·420752	9·569488	9·967826	9·601663	10·398337	13
48	9·550359	9·970731	9·579629	10·420371	9·569804	9·967775	9·602029	10·397971	12
49	9·550692	9·970683	9·580009	10·419991	9·570120	9·967725	9·602395	10·397605	11
50	9·551024	9·970635	9·580389	10·419611	9·570435	9·967674	9·602761	10·397239	10
51	9·551356	9·970586	9·580769	10·419231	9·570751	9·967624	9·603127	10·396873	9
52	9·551687	9·970538	9·581149	10·418851	9·571066	9·967573	9·603493	10·396507	8
53	9·552018	9·970490	9·581528	10·418472	9·571380	9·967522	9·603858	10·396142	7
54	9·552349	9·970442	9·581907	10·418093	9·571695	9·967471	9·604223	10·395777	6
55	9·552680	9·970394	9·582286	10·417714	9·572009	9·967421	9·604588	10·395412	5
56	9·553010	9·970345	9·582665	10·417335	9·572323	9·967370	9·604953	10·395047	4
57	9·553341	9·970297	9·583044	10·416956	9·572636	9·967319	9·605317	10·394683	3
58	9·553670	9·970249	9·583422	10·416578	9·572950	9·967268	9·605682	10·394318	2
59	9·554000	9·970200	9·583800	10·416200	9·573263	9·967217	9·606046	10·393954	1
	9·554329	9·970152	9·584177	10·415823	9·573575	9·967166	9·606410	10·393590	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine.	Cotan.	Tang.	′
	69 Deg.				68 Deg.				

	22 Deg.				23 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·573575	9·967166	9·606410	10·393590	9·591878	9·964026	9·627852	10·372148	60
1	9·573888	9·967115	9·606773	10·393227	9·592176	9·963972	9·628205	10·371797	59
2	9·574200	9·967064	9·607137	10·392863	9·592473	9·963919	9·628554	10·371446	58
3	9·574512	9·967013	9·607500	10·392500	9·592770	9·963865	9·628905	10·371095	57
4	9·574824	9·966961	9·607863	10·392137	9·593067	9·963811	9·629255	10·370745	56
5	9·575136	9·966910	9·608225	10·391775	9·593363	9·963757	9·629606	10·370394	55
6	9·575447	9·966859	9·608588	10·391412	9·593659	9·963704	9·629956	10·370044	54
7	9·575758	9·966808	9·608950	10·391050	9·593955	9·963650	9·630306	10·369694	53
8	9·576069	9·966756	9·609312	10·390688	9·594251	9·963596	9·630656	10·369344	52
9	9·576379	9·966705	9·609674	10·390326	9·594547	9·963542	9·631005	10·368995	51
10	9·576689	9·966653	9·610036	10·389964	9·594842	9·963488	9·631355	10·368645	50
11	9·576999	9·966602	9·610397	10·389603	9·595137	9·963434	9·631704	10·368296	49
12	9·577309	9·966550	9·610759	10·389241	9·595432	9·963379	9·632053	10·367947	48
13	9·577618	9·966499	9·611120	10·388880	9·595727	9·963325	9·632402	10·367598	47
14	9·577927	9·966447	9·611480	10·388520	9·596021	9·963271	9·632750	10·367250	46
15	9·578236	9·966395	9·611841	10·388159	9·596315	9·963217	9·633099	10·366901	45
16	9·578545	9·966344	9·612201	10·387799	9·596609	9·963163	9·633447	10·366553	44
17	9·578853	9·966292	9·612561	10·387439	9·596903	9·963108	9·633795	10·366205	43
18	9·579162	9·966240	9·612921	10·387079	9·597196	9·963054	9·634143	10·365857	42
19	9·579470	9·966188	9·613281	10·386719	9·597490	9·962999	9·634490	10·355510	41
20	9·579777	9·966136	9·613641	10·386359	9·597783	9·962945	9·634838	10·365162	40
21	9·580085	9·966085	9·614000	10·386000	9·598075	9·962890	9·635185	10·364815	39
22	9·580392	9·966033	9·614359	10·385641	9·598368	9·962836	9·635532	10·364468	38
23	9·580699	9·965981	9·614718	10·385282	9·598660	9·962781	9·635879	10·364121	37
24	9·581005	9·965929	9·615077	10·384923	9·598952	9·962727	9·636226	10·363774	36
25	9·581312	9·965876	9·615435	10·384565	9·599244	9·962672	9·636572	10·363428	35
26	9·581618	9·965824	9·615793	10·384207	9·599536	9·962617	9·636919	10·363081	34
27	9·581924	9·965772	9·616151	10·383849	9·599827	9·962562	9·637265	10·362735	33
28	9·582229	9·965720	9·616509	10·383491	9·600118	9·962508	9·637611	10·362389	32
29	9·582535	9·965668	9·616867	10·383133	9·600409	9·962453	9·637956	10·362044	31
30	9·582840	9·965615	9·617224	10·382776	9·600700	9·962398	9·638302	10·361698	30
31	9·583145	9·965563	9·617582	10·382418	9·600990	9·962343	9·638647	10·361353	29
32	9·583449	9·965511	9·617939	10·382061	9·601280	9·962288	9·638992	10·361008	28
33	9·583754	9·965458	9·618295	10·381705	9·601570	9·962233	9·639337	10·360663	27
34	9·584058	9·965406	9·618652	10·381348	9·601860	9·962178	9·639682	10·360318	26
35	9·584361	9·965353	9·619008	10·380992	9·602150	9·962123	9·640027	10·359973	25
36	9·584665	9·965301	9·619364	10·380636	9·602439	9·962067	9·640371	10·359629	24
37	9·584968	9·965248	9·619720	10·380280	9·602728	9·962012	9·640716	10·359284	23
38	9·585272	9·965195	9·620076	10·379924	9·603017	9·961957	9·641060	10·358940	22
39	9·585574	9·965143	9·620432	10·379568	9·603305	9·961902	9·641404	10·358596	21
40	9·585877	9·965090	9·620787	10·379213	9·603594	9·961846	9·641747	10·358253	20
41	9·586179	9·965037	9·621142	10·378858	9·603882	9·961791	9·642091	10·357909	19
42	9·586482	9·964984	9·621497	10·378503	9·604170	9·961735	9·642434	10·357566	18
43	9·586783	9·964931	9·621852	10·378148	9·604457	9·961680	9·642777	10·357223	17
44	9·587085	9·964879	9·622207	10·377793	9·604745	9·961624	9·643120	10·356880	16
45	9·587386	9·964826	9·622561	10·377439	9·605032	9·961569	9·643463	10·356537	15
46	9·587688	9·964773	9·622915	10·377085	9·605319	9·961513	9·643806	10·356194	14
47	9·587989	9·964720	9·623269	10·376731	9·605606	9·961458	9·644148	10·355852	13
48	9·588289	9·964666	9·623623	10·376377	9·605892	9·961402	9·644490	10·355510	12
49	9·588590	9·964613	9·623976	10·376024	9·606179	9·961346	9·644832	10·355168	11
50	9·588890	9·964560	9·624330	10·375670	9·606465	9·961290	9·645174	10·354826	10
51	9·589190	9·964507	9·624683	10·375317	9·606751	9·961235	9·645516	10·354484	9
52	9·589489	9·964454	9·625036	10·374964	9·607036	9·961179	9·645857	10·354143	8
53	9·589789	9·964400	9·625388	10·374612	9·607322	9·961123	9·646199	10·353801	7
54	9·590088	9·964347	9·625741	10·374259	9·607607	9·961067	9·646540	10·353460	6
55	9·590387	9·964294	9·626093	10·373907	9·607892	9·961011	9·646881	10·353119	5
56	9·590686	9·964240	9·626445	10·373555	9·608177	9·960955	9·647222	10·352778	4
57	9·590984	9·964187	9·626797	10·373203	9·608461	9·960899	9·647562	10·352438	3
58	9·591282	9·964133	9·627149	10·372851	9·608745	9·960843	9·647903	10·352097	2
59	9·591580	9·964080	9·627501	10·372499	9·609029	9·960786	9·648243	10·351757	1
60	9·591878	9·964026	9·627852	10·372148	9·609313	9·960730	9·648583	10·351417	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	
	67 Deg.				66 Deg.				

	24 Deg.				25 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·609313	9·960730	9·648583	10·351417	9·625948	9·957276	9·668673	10·331327	60
1	9·609597	9·960674	9·648923	10·351077	9·626219	9·957217	9·669002	10·330998	59
2	9·609880	9·960618	9·649990	10·350737	9·626490	9·957158	9·669332	10·330668	58
3	9·610164	9·960561	9·649602	10·350398	9·626760	9·957099	9·669661	10·330339	57
4	9·610447	9·960505	9·649942	10·350058	9·627030	9·957040	9·669991	10·330009	56
5	9·610729	9·960448	9·650281	10·349719	9·627300	9·956981	9·670320	10·329680	55
6	9·611012	9·960392	9·650620	10·349380	9·627570	9·956921	9·670649	10·329351	54
7	9·611294	9·960335	9·650959	10·349041	9·627840	9·956862	9·670977	10·329023	53
8	9·611576	9·960279	9·651297	10·348703	9·628109	9·956803	9·671306	10·328694	52
9	9·611858	9·960222	9·651636	10·348364	9·628378	9·956744	9·671635	10·328365	51
10	9·612140	9·960165	9·651974	10·348026	9·628647	9·956684	9·671963	10·328037	50
11	9·612421	9·960109	9·652312	10·347688	9·628916	9·956625	9·672291	10·327709	49
12	9·612702	9·960052	9·652650	10·347350	9·629185	9·956566	9·672619	10·327381	48
13	9·612983	9·959995	9·652988	10·347012	9·629453	9·956506	9·672947	10·327053	47
14	9·613264	9·959938	9·653326	10·346674	9·629721	9·956447	9·673274	10·326726	46
15	9·613545	9·959882	9·653663	10·346337	9·629989	9·956387	9·673602	10·326398	45
16	9·613825	9·959825	9·654000	10·346000	9·630257	9·956327	9·673929	10·326071	44
17	9·614105	9·959768	9·654337	10·345663	9·630524	9·956268	9·674257	10·325743	43
18	9·614385	9·959711	9·654674	10·345326	9·630792	9·956208	9·674584	10·325416	42
19	9·614665	9·959654	9·655011	10·344989	9·631059	9·956148	9·674911	10·325089	41
20	9·614944	9·959596	9·655348	10·344652	9·631326	9·956089	9·675237	10·324763	40
21	9·615223	9·959539	9·655684	10·344316	9·631593	9·956029	9·675564	10·324436	39
22	9·615502	9·959489	9·656020	10·343980	9·651859	9·955969	9·675890	10·324110	38
23	9·615781	9·959425	9·656356	10·343644	9·632125	9·955909	9·676217	10·323783	37
24	9·616060	9·959368	9·656692	10·343308	9·632392	9·955849	9·676543	10·323457	36
25	9·616358	9·959310	9·657028	10·342972	9·632658	9·955789	9·676869	10·323131	35
26	9·616616	9·959253	9·657364	10·342636	9·632923	9·955729	9·677194	10·322806	34
27	9·616894	9·959195	9·657699	10·342301	9·633189	9·955669	9·677520	10·322480	33
28	9·617172	9·959138	9·658034	10·341966	9·633454	9·955609	9·677846	10·322154	32
29	9·617450	9·959080	9·658369	10·341631	9·633719	9·955548	9·678171	10·321829	31
30	9·617727	9·959023	9·658704	10·341296	9·633984	9·955488	9·678496	10·321504	30
31	9·618004	9·958965	9·659039	10·340961	9·634249	9·955428	9·678821	10·321179	29
32	9·618281	9·958908	9·659373	10·340327	9·634514	9·955368	9·679146	10·320854	28
33	9·618558	9·958850	9·659708	10·340292	9·634778	9·955307	9·679471	10·320529	27
34	9·618834	9·958792	9·660042	10·339958	9·635042	9·955247	9·679795	10·320205	26
35	9·619110	9·958734	9·660376	10·339624	9·635306	9·955186	9·680120	10·319880	25
36	9·619386	9·958677	9·660710	10·339290	9·635570	9·955126	9·680444	10·319556	24
37	9·619662	9·958619	9·661043	10·338957	9·635834	9·955065	9·680768	10·319232	23
38	9·619938	9·958561	9·661377	10·338623	9·636097	9·955005	9·681092	10·318908	22
39	9·620213	9·958503	9·661710	10·338290	9·636360	9·954944	9·681416	10·318584	21
40	9·620488	9·958445	9·662043	10·337957	9·636623	9·954883	9·681740	10·318260	20
41	9·620763	9·958387	9·662376	10·337624	9·636886	9·954823	9·682063	10·317937	19
42	9·621038	9·958329	9·662709	10·337291	9·637148	9·954762	9·682387	10·317613	18
43	9·621313	9·958271	9·663042	10·336958	9·637411	9·954701	9·682710	10·317290	17
44	9·621587	9·958213	9·663375	10·336625	9·637673	9·954640	9·683033	10·316967	16
45	9·621861	9·958154	9·663707	10·336293	9·637935	9·954579	9·683356	10·316644	15
46	9·622135	9·958096	9·664039	10·335961	9·638197	9·954518	9·683679	10·316321	14
47	9·622409	9·958038	9·664371	10·335629	9·638458	9·954457	9·684001	10·315999	13
48	9·622682	9·957979	9·664703	10·335297	9·638720	9·954396	9·684324	10·315676	12
49	9·622956	9·957921	9·665035	10·334965	9·638981	9·954335	9·684646	10·315354	11
50	9·623229	9·957863	9·665366	10·334634	9·639242	9·954274	9·684968	10·315032	10
51	9·623502	9·957804	9·665698	10·334302	9·639503	9·954213	9·685290	10·314710	9
52	9·623774	9·957746	9·666029	10·333971	9·639764	9·954152	9·685612	10·314388	8
53	9·624047	9·957687	9·666360	10·333640	9·640024	9·954090	9·685934	10·314066	7
54	9·624319	9·957628	9·666691	10·333309	9·640284	9·954029	9·686255	10·313745	6
55	9·624591	9·957570	9·667021	10·332979	9·640544	9·953968	9·686577	10·313423	5
56	9·624863	9·957511	9·667352	10·332648	9·640804	9·953906	9·686898	10·313102	4
57	9·625135	9·957452	9·667682	10·332318	9·641064	9·953845	9·687219	10·312781	3
58	9·625406	9·957393	9·668013	10·331987	9·641324	9·953783	9·687540	10·312460	2
59	9·625677	9·957335	9·668343	10·331657	9·641583	9·953722	9·687861	10·312139	1
60	9·625948	9·957276	9·668673	10·331327	9·641842	9·953660	9·688182	10·311818	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	65 Deg.				64 Deg.				

	26 Deg.				27 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·641842	9·953660	9·688182	10·311818	9·657047	9·949881	9·707166	10·292834	60
1	9·642101	9·953599	9·688502	10·311498	9·657295	9·949816	9·707478	10·292522	59
2	9·642360	9·953537	9·688823	10·311177	9·657542	9·949752	9·707790	10·292210	58
3	9·642618	9·953475	9·689143	10·310857	9·657790	9·949688	9·708102	10·291898	57
4	9·642877	9·953413	9·689463	10·310537	9·658037	9·949623	9·708414	10·291586	56
5	9·643135	9·953352	9·689783	10·310217	9·658284	9·949558	9·708726	10·291274	55
6	9·643393	9·953290	9·690103	10·309897	9·658531	9·949494	9·709037	10·290963	54
7	9·643650	9·953228	9·690423	10·309577	9·658778	9·949429	9·709349	10·290651	53
8	9·643908	9·953166	9·690742	10·309258	9·659025	9·949364	9·709660	10·290340	52
9	9·644165	9·953104	9·691062	10·308938	9·659271	9·949300	9·709971	10·290029	51
10	9·644423	9·953042	9·691381	10·308619	9·659517	9·949235	9·710282	10·289718	50
11	9·644680	9·952980	9·691700	10·308300	9·659763	9·949170	9·710593	10·289407	49
12	9·644936	9·952918	9·692019	10·307981	9·660009	9·949105	9·710904	10·289096	48
13	9·645193	9·952855	9·692338	10·307662	9·660255	9·949040	9·711215	10·288785	47
14	9·645450	9·952793	9·692656	10·307344	9·660501	9·948975	9·711525	10·288475	46
15	9·645706	9·952731	9·692975	10·307025	9·660746	9·948910	9·711836	10·288164	45
16	9·645962	9·952669	9·693293	10·306707	9·660991	9·948845	9·712146	10·287854	44
17	9·646218	9·952606	9·693612	10·306388	9·661236	9·948780	9·712456	10·287544	43
18	9·646474	9·952544	9·693930	10·306070	9·661481	9·948715	9·712766	10·287234	42
19	9·646729	9·952481	9·694248	10·305752	9·661726	9·948650	9·713076	10·286924	41
20	9·646984	9·952419	9·694566	10·305434	9·661970	9·948584	9·713386	10·286614	40
21	9·647240	9·952356	9·694883	10·305117	9·662214	9·948519	9·713696	10·286304	39
22	9·647494	9·952294	9·695201	10·304799	9·662459	9·948454	9·714005	10·285995	38
23	9·647749	9·952231	9·695518	10·304482	9·662703	9·948388	9·714314	10·285686	37
24	9·648004	9·952168	9·695836	10·304164	9·662946	9·948323	9·714624	10·285376	36
25	9·648258	9·952106	9·696153	10·303847	9·663190	9·948257	9·714933	10·285067	35
26	9·648512	9·952043	9·696470	10·303530	9·663433	9·948192	9·715242	10·284758	34
27	9·648766	9·951980	9·696787	10·303213	9·663677	9·948126	9·715551	10·284449	33
28	9·649020	9·951917	9·697103	10·302897	9·663920	9·948060	9·715860	12·284140	32
29	9·649274	9·951854	9·697420	10·302580	9·664163	9·947995	9·716168	10·283832	31
30	9·649527	9·951791	9·697736	10·302264	9·664406	9·947929	9·716477	10·283523	30
31	9·649781	9·951728	9·698053	10·301947	9·664648	9·947863	9·716785	10·28[illegible]215	29
32	9·650034	9·951665	9·698369	10·301631	9·664891	9·947797	9·717093	10·282907	28
33	9·650287	9·951602	9·698685	10·301315	9·665133	9·947731	9·717401	10·282599	27
34	9·650539	9·951539	9·699001	10·300999	9·665375	9·947665	9·717709	10·282291	26
35	9·650792	9·951476	9·699316	10·300684	9·665617	9·947600	9·718017	10·281983	25
36	9·651044	9·951412	9·699632	10·300368	9·665859	9·947533	9·718325	10·281675	24
37	9·651297	9·951349	9·699947	10·300053	9·666100	9·947467	9·718633	10·281367	23
38	9·651549	9·951286	9·700263	10·299737	9·666342	9·947401	9·718940	10·281060	22
39	9·651800	9·951222	9·700578	10·299422	9·666583	9·947335	9·719248	10·280752	21
40	9·652052	9·951159	9·700893	10·299107	9·666824	9·947269	9·719555	10·280445	20
41	9·652304	9·951096	9·701208	10·298792	9·667065	9·947203	9·719862	10·280138	19
42	9·652555	9·951032	9·701523	10·298477	9·667305	9·947136	9·720169	10·279831	18
43	9·652806	9·950968	9·701837	10·298163	9·667546	9·947070	9·720476	10·279524	17
44	9·653057	9·950905	9·702152	10·297848	9·667786	9·947004	9·720783	10·279217	16
45	9·653308	9·950841	9·702466	10·297534	9·668027	9·946937	9·721089	10·278911	15
46	9·653558	9·950778	9·702781	10·297219	9·668267	9·946871	9·721396	10·278604	14
47	9·653808	9·950714	9·703095	10·296905	9·668506	9·946804	9·721702	10·278298	13
48	9·654059	9·950650	9·703409	10·296591	9·668746	9·946738	9·722009	10·277991	12
49	9·654309	9·950586	9·703722	10·296278	9·668986	9·946671	9·722315	10·277685	11
50	9·654558	9·950522	9·704036	10·295964	9·669225	9·946604	9·722621	10·277379	10
51	9·654808	9·950458	9·704350	10·295650	9·669464	9·946538	9·722927	10·277073	9
52	9·655058	9·950394	9·704663	10·295337	9·669703	9·946471	9·723232	10·276768	8
53	9·655307	9·950330	9·704976	10·295024	9·669942	9·946404	9·723538	10·276462	7
54	9·655556	9·950266	9·705290	10·294710	9·670181	9·946337	9·723844	10·276156	6
55	9·655805	9·950202	9·705603	10·294397	9·670419	9·946270	9·724149	10·275851	5
56	9·656054	9·950138	9·705916	10·294084	9·670658	9·946203	9·724454	10·275546	4
57	9·656302	9·950074	9·706228	10·293772	9·670896	9·946136	9·724760	10·275240	3
58	9·656561	9·950010	9·706541	10·293459	9·671134	9·946069	9·725065	10·274935	2
59	9·656799	9·949945	9·706854	10·293146	9·671372	9·946002	9·725370	10·274630	1
60	9·657047	9·949881	9·707166	10·292834	9·671609	9·945935	9·725674	10·274326	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	63 Deg.				62 Deg.				

	28 Deg.				29 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·671609	9·945935	9·725674	10·274326	9·685571	9·941819	9·743752	10·256248	60
1	9·671847	9·945868	9·725979	10·274021	9·685799	9·941749	9·744050	10·255950	59
2	9·672084	9·945800	9·726284	10·273716	9·686027	9·941679	9·744348	10·255652	58
3	9·672321	9·945733	9·726588	10·273412	9·686254	9·941609	9·744645	10·255355	57
4	9·672558	9·945666	9·726892	10·273108	9·686482	9·941539	9·744943	10·255057	56
5	9·67279[illegible]	9·945598	9·727197	10·272803	9·686709	9·941469	9·745240	10·254760	55
6	9·673032	9·945531	9·727501	10·272499	9·686936	9·941398	9·745538	10·254462	54
7	9·673268	9·945464	9·727805	10·272195	9·687163	9·941328	9·745835	10·254165	53
8	9·673505	9·945396	9·728109	10·271891	9·687389	9·941258	9·746132	10·253868	52
9	9·673741	9·945328	9·728412	10·271588	9·687616	9·941187	9·746429	10·253571	51
10	9·673977	9·945261	9·728716	10·271284	9·687843	9·941117	9·746726	10·253274	50
11	9·674213	9·945193	9·729020	10·270980	9·688069	9·941046	9·747023	10·252977	49
12	9·674448	9·945125	9·729323	10·270677	9·688295	9·940975	9·747319	10·252681	48
13	9·674684	9·945058	9·729626	10·270374	9·688521	9·940905	9·747616	10·252384	47
14	9·674919	9·944990	9·729929	10·270071	9·688747	9·940834	9·747913	10·252087	46
15	9·675155	9·944922	9·730233	10·269767	9·688972	9·940763	9·748209	10·251791	45
16	9·675390	9·944854	9·730535	10·269465	9·689198	9·940693	9·748505	10·251495	44
17	9·675624	9·944786	9·730838	10·269162	9·689423	9·940622	9·748801	10·251199	43
18	9·675859	9·944718	9·731141	10·268859	9·689648	9·940551	9·749097	10·250903	42
19	9·676094	9·944650	9·731444	10·268556	9·689873	9·940480	9·749393	10·250607	41
20	9·676328	9·944582	9·731746	10·268254	9·690098	9·940409	9·749689	10·250311	40
21	9·676562	9·944514	9·732048	10·267952	9·690323	9·940338	9·749985	10·250015	39
22	9·676796	9·944446	9·732351	10·267649	9·690548	9·940267	9·750281	10·249719	38
23	9·677030	9·944377	9·732653	10·267347	9·690772	9·940196	9·750576	10·249424	37
24	9·677264	9·944309	9·732955	10·267045	9·690996	9·940125	9·750872	10·249128	36
25	9·677498	9·944241	9·733257	10·266743	9·691220	9·940054	9·751167	10·248833	35
26	9·677731	9·944172	9·733558	10·266442	9·691444	9·939982	9·751462	10·248538	34
27	9·677964	9·944104	9·733860	10·266140	9·691668	9·939911	9·751757	10·248243	33
28	9·678197	9·944036	9·734162	10·265838	9·691892	9·939840	9·752052	10·247948	32
29	9·678430	9·943967	9·734463	10·265537	9·692115	9·939768	9·752347	10·247653	31
30	9·678663	9·943899	9·734764	10·265236	9·692339	9·939697	9·752642	10·247358	30
31	9·678895	9·943830	9·735066	10·264934	9·692562	9·939625	9·752937	10·247063	29
32	9·679128	9·943761	9·735367	10·264633	9·692785	9·939554	9·753231	10·246769	28
33	9·679360	9·943693	9·735668	10·264332	9·693008	9·939482	9·753526	10·246474	27
34	9·679592	9·943624	9·735969	10·264031	9·69323[illegible]	9·939410	9·753820	10·246180	26
35	9·679824	9·943555	9·736269	10·263731	9·693453	9·939339	9·754115	10·245885	25
36	9·680056	9·943486	9·736570	10·263430	9·693676	9·939267	9·754409	10·245591	24
37	9·680288	9·943417	9·736870	10·263130	9·693898	9·939195	9·754703	10·245297	23
38	9·680519	9·943348	9·737171	10·262829	9·694120	9·939123	9·754997	10·245003	22
39	9·680750	9·943279	9·737471	10·262529	9·694342	9·939052	9·755291	10·244709	21
40	9·680982	9·943210	9·737771	10·262229	9·694564	9·938980	9·755585	10·244415	20
41	9·681213	9·943141	9·738071	10·261929	9·694786	9·938908	9·755878	10·244122	19
42	9·681443	9·943072	9·738371	10·261629	9·695007	9·938836	9·756172	10·243828	18
43	9·681674	9·943003	9·738671	10·261329	9·695229	9·938763	9·756465	10·243535	17
44	9·681905	9·942934	9·738971	10·261029	9·695450	9·938691	9·756759	10·243241	16
45	9·682135	9·942864	9·739271	10·260729	9·695671	9·938619	9·757052	10·242948	15
46	9·682365	9·942795	9·739570	10·260430	9·695892	9·938547	9·757345	10·242655	14
47	9·682595	9·942726	9·739870	10·260130	9·696113	9·938475	9·757638	10·242362	13
48	9·682825	9·942656	9·740169	10·259831	9·696334	9·938402	9·757931	10·242069	12
49	9·683055	9·942587	9·740468	10·259532	9·696554	9·938330	9·758224	10·241776	11
50	9·683284	9·942517	9·740767	10·259233	9·696775	9·938258	9·758517	10·241483	10
51	9·683514	9·942448	9·741066	10·258934	9·696995	9·938185	9·758810	10·241190	9
52	9·683743	9·942378	9·741365	10·258635	9·697215	9·938113	9·759102	10·240898	8
53	9·683972	9·942308	9·741664	10·258336	9·697435	9·938040	9·759395	10·240605	7
54	9·684201	9·942239	9·741962	10·258038	9·697654	9·937967	9·759687	10·240313	6
55	9·684430	9·942169	9·742261	10·257739	9·697874	9·937895	9·759979	10·240021	5
56	9·684658	9·942099	9·742559	10·257441	9·698094	9·937822	9·760272	10·239728	4
57	9·684887	9·942029	9·742858	10·257142	9·698313	9·937749	9·760564	10·239436	3
58	9·685115	9·941959	9·743156	10·256844	9·698532	9·937676	9·760856	10·239144	2
59	9·685343	9·941889	9·743454	10·256546	9·698751	9·937604	9·761148	10·238852	1
60	9·685571	9·941819	9·743752	10·256248	9·698970	9·937531	9·761439	10·238561	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	61 Deg.				60 Deg.				

	30 Deg.				31 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·698970	9·937531	9·761439	10·238561	9·711839	9·933066	9·778774	10·221226	60
1	9·699189	9·937458	9·761731	10·238269	9·712050	9·932990	9·779060	10·220940	59
2	9·699407	9·937385	9·762023	10·237977	9·712260	9·932914	9·779346	10·220654	58
3	9·699626	9·937312	9·762314	10·237686	9·712469	9·932838	9·779632	10·220368	57
4	9·699844	9·937238	9·762606	10·237394	9·712679	9·932762	9·779918	10·220082	56
5	9·700062	9·937165	9·762897	10·237103	9·712889	9·932685	9·780203	10·219797	55
6	9·700280	9·937092	9·763188	10·236812	9·713098	9·932609	9·780489	10·219511	54
7	9·700498	9·937019	9·763479	10·236521	9·713308	9·932533	9·780775	10·219225	53
8	9·700716	9·936946	9·763770	10·236230	9·713517	9·932457	9·781060	10·218940	52
9	9·700933	9·936872	9·764061	10·235939	9·713726	9·932380	9·781346	10·218654	51
10	9·701151	9·936799	9·764352	10·235648	9·713935	9·932304	9·781631	10·218369	50
11	9·701368	9·936725	9·764643	10·235357	9·714144	9·932228	9·781916	10·218084	49
12	9·701585	9·936652	9·764933	10·235067	9·714352	9·932151	9·782201	10·217799	48
13	9·701802	9·936578	9·765224	10·234776	9·714561	9·932075	9·782486	10·217514	47
14	9·702019	9·936505	9·765514	10·234486	9·714769	9·931998	9·782771	10·217229	46
15	9·702236	9·936431	9·765805	10·234195	9·714978	9·931921	9·783056	10·216944	45
16	9·702452	9·936357	9·766095	10·233905	9·715186	9·931845	9·783341	10·216659	44
17	9·702669	9·936284	9·766385	10·233615	9·715394	9·931768	9·783626	10·216374	43
18	9·702885	9·936210	9·766675	10·233325	9·715602	9·931691	9·783910	10·216090	42
19	9·703101	9·936136	9·766965	10·233035	9·715809	9·931614	9·784195	10·215805	41
20	9·703317	9·936062	9·767255	10·232745	9·716017	9·931537	9·784479	10·215521	40
21	9·703533	9·935988	9·767545	10·232455	9·716224	9·931460	9·784764	10·215236	39
22	9·703749	9·935914	9·76783-	10·232166	9·716432	9·931383	9·785048	10·214952	38
23	9·703964	9·935840	9·768124	10·231876	9·716639	9·931306	9·785332	10·214668	37
24	9·704179	9·935766	9·768414	10·231586	9·716846	9·931229	9·785616	10·214384	36
25	9·704395	9·935692	9·768703	10·231297	9·717053	9·931152	9·785900	10·214100	35
26	9·704610	9·935618	9·768992	10·231008	9·717259	9·931075	9·786184	10·213816	34
27	9·704825	9·935543	9·769281	10·230719	9·717466	9·930998	9·786468	10·213532	33
28	9·705040	9·935469	9·769571	10·230429	9·717673	9·930921	9·786752	10·213248	32
29	9·705254	9·935395	9·769860	10·230140	9·717879	9·930843	9·787036	10·212964	31
30	9·705469	9·935320	9·770148	10·229852	9·718085	9·939766	9·787319	10·212681	30
31	9·705683	9·935246	9·770437	10·229563	9·718291	9·930688	9·787603	10·212397	29
32	9·705898	9·935171	9·770726	10·229274	9·718497	9·930611	9·787886	10·212114	28
33	9·706112	9·935097	9·771015	10·228985	9·718703	9·930533	9·788170	10·211830	27
34	9·706326	9·935022	9·771303	10·228697	9·718909	9·930456	9·788453	10·211547	26
35	9·706539	9·934948	9·771592	10·228408	9·719114	9·930378	9·788736	10·211264	25
36	9·706753	9·934873	9·771880	10·228120	9·719320	9·930300	9·789019	10·210981	24
37	9·706967	9·934798	9·772168	10·227832	9·719525	9·930223	9·789302	10·210698	23
38	9·707180	9·934723	9·772457	10·227543	9·719730	9·930145	9·789585	10·210415	22
39	9·707393	9·934649	9·772745	10·227255	9·719935	9·930067	9·789868	10·210132	21
40	9·707606	9·934574	9·773033	10·226967	9·720140	9·929989	9·790151	10·209849	20
41	9·707819	9·934499	9·773321	10·226679	9·720345	9·929911	9·790434	10·209566	19
42	9·708032	9·934424	9·773608	10·226392	9·720549	9·929833	9·790716	10·209284	18
43	9·708245	9·934349	9·773896	10·226104	9·720754	9·929755	9·790999	10·209001	17
44	9·708458	9·934274	9·774184	10·225816	9·720958	9·929677	9·791281	10·208719	16
45	9·708670	9·934199	9·774971	10·225529	9·721162	9·929599	9·791563	10·208437	15
46	9·708882	9·934123	9·774759	10·225241	9·721366	9·929521	9·791846	10·208154	14
47	9·709094	9·934048	9·775046	10·224954	9·721570	9·929442	9·792128	10·207872	13
48	9·709306	9·933973	9·775333	10·224667	9·721774	9·929364	9·792410	10·207590	12
49	9·709518	9·933898	9·775621	10·224379	9·721978	9·929286	9·792692	10·207308	11
50	9·709730	9·933822	9·775908	10·224092	9·722181	9·929207	9·792974	10·207026	10
51	9·709941	9·933747	9·776195	10·223805	6·722385	9·929129	9·793256	10·206744	9
52	9·710153	9·933671	9·776482	10·223518	9·722588	9·929050	9·793538	10·206462	8
53	9·710364	9·933596	9·776769	10·223232	9·722791	9·928972	9·793819	10·206181	7
54	9·710575	9·933520	9·777055	10·222945	9·722994	9·928853	9·794101	10·205899	6
55	9·710786	9·933445	9·777342	10·222658	9·723197	9·928815	9·794383	10·205617	5
56	9·710997	9·933369	9·777628	10·222372	9·723400	9·928736	9·794664	10·205336	4
57	9·711208	9·933292	9·777915	10·222085	9·723603	9·928657	9·794946	10·205054	3
58	9·711419	9·933217	9·778201	10·221799	9·723805	9·928578	9·795227	10·204773	2
59	9·711629	9·933141	9·778488	10·221512	9·724007	9·928499	9·795508	10·204492	1
60	9·711839	9·93306	9·778774	10·221226	9·724210	9·928420	9·795789	10·204211	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	59 Deg.				58 Deg.				

	28 Deg.				29 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·671609	9·945935	9·725674	10·274326	9·685571	9·941819	9·743752	10·256248	60
1	9·671847	9·945868	9·725979	10·274021	9·685799	9·941749	9·744050	10·255950	59
2	9·672084	9·945800	9·726284	10·273716	9·686027	9·941679	9·744348	10·255652	58
3	9·672321	9·945733	9·726588	10·273412	9·686254	9·941609	9·744645	10·255355	57
4	9·672558	9·945666	9·726892	10·273108	9·686482	9·941539	9·744943	10·255057	56
5	9·672795	9·945598	9·727197	10·272803	9·686709	9·941469	9·745240	10·254760	55
6	9·673032	9·945531	9·727501	10·272499	9·686936	9·941398	9·745538	10·254462	54
7	9·673268	9·945464	9·727805	10·272195	9·687163	9·941328	9·745835	10·254165	53
8	9·673505	9·945396	9·728109	10·271891	9·687389	9·941258	9·746132	10·253868	52
9	9·673741	9·945328	9·728412	10·271588	9·687616	9·941187	9·746429	10·253571	51
10	9·673977	9·945261	9·728716	10·271284	9·687843	9·941117	9·746726	10·253274	50
11	9·674213	9·945193	9·729020	10·270980	9·688069	9·941046	9·747023	10·252977	49
12	9·674448	9·945125	9·729323	10·270677	9·688295	9·940975	9·747319	10·252681	48
13	9·674684	9·945058	9·729626	10·270374	9·688521	9·940905	9·747616	10·252384	47
14	9·674919	9·944990	9·729929	10·270071	9·688747	9·940834	9·747913	10·252087	46
15	9·675155	9·944922	9·730233	10·269767	9·688972	9·940763	9·748209	10·251791	45
16	9·675390	9·944854	9·730535	10·269465	9·689198	9·940693	9·748505	10·251495	44
17	9·675624	9·944786	9·730838	10·269162	9·689423	9·940622	9·748801	10·251199	43
18	9·675859	9·944718	9·731141	10·268859	9·689648	9·940551	9·749097	10·250903	42
19	9·676094	9·944650	9·731444	10·268556	9·689873	9·940480	9·749393	10·250607	41
20	9·676328	9·944582	9·731746	10·268254	9·690098	9·940409	9·749689	10·250311	40
21	9·676562	9·944514	9·732048	10·267952	9·690323	9·940338	9·749985	10·250015	39
22	9·676796	9·944446	9·732351	10·267649	9·690548	9·940267	9·750281	10·249719	38
23	9·677030	9·944377	9·732653	10·267347	9·690772	9·940196	9·750576	10·249424	37
24	9·677264	9·944309	9·732955	10·267045	9·690996	9·940125	9·750872	10·249128	36
25	9·677498	9·944241	9·733257	10·266743	9·691220	9·940054	9·751167	10·248833	35
26	9·677731	9·944172	9·733558	10·266442	9·691444	9·939982	9·751462	10·248538	34
27	9·677964	9·944104	9·733860	10·266140	9·691668	9·939911	9·751757	10·248243	33
28	9·678197	9·944036	9·734162	10·265838	9·691892	9·939840	9·752052	10·247948	32
29	9·678430	9·943967	9·734463	10·265537	9·692115	9·939768	9·752347	10·247653	31
30	9·678663	9·943899	9·734764	10·265236	9·692339	9·939697	9·752642	10·247358	30
31	9·678895	9·943830	9·735066	10·264934	9·692562	9·939625	9·752937	10·247063	29
32	9·679128	9·943761	9·735367	10·264633	9·692785	9·939554	9·753231	10·246769	28
33	9·679360	9·943693	9·735668	10·264332	9·693008	9·939482	9·753526	10·246474	27
34	9·679592	9·943624	9·735969	10·264031	9·69323[illegible]	9·939410	9·753820	10·246180	26
35	9·679824	9·943555	9·736269	10·263731	9·693453	9·939339	9·754115	10·245885	25
36	9·680056	9·943486	9·736570	10·263430	9·693676	9·939267	9·754409	10·245591	24
37	9·680288	9·943417	9·736870	10·263130	9·693898	9·939195	9·754703	10·245297	23
38	9·680519	9·943348	9·737171	10·262829	9·694120	9·939123	9·754997	10·245003	22
39	9·680750	9·943279	9·737471	10·262529	9·694342	9·939052	9·755291	10·244709	21
40	9·680982	9·943210	9·737771	10·262229	9·694564	9·938980	9·755585	10·244415	20
41	9·681213	9·943141	9·738071	10·261929	9·694786	9·938908	9·755878	10·244122	19
42	9·681443	9·943072	9·738371	10·261629	9·695007	9·938836	9·756172	10·243828	18
43	9·681674	9·943003	9·738671	10·261329	9·695229	9·938763	9·756465	10·243535	17
44	9·681905	9·942934	9·738971	10·261029	9·695450	9·938691	9·756759	10·243241	16
45	9·682135	9·942864	9·739271	10·260729	9·695671	9·938619	9·757052	10·242948	15
46	9·682365	9·942795	9·739570	10·260430	9·695892	9·938547	9·757345	10·242655	14
47	9·682595	9·942726	9·739870	10·260130	9·696113	9·938475	9·757638	10·242362	13
48	9·682825	9·942656	9·740169	10·259831	9·696334	9·938402	9·757931	10·242069	12
49	9·683055	9·942587	9·740468	10·259532	9·696554	9·938330	9·758224	10·241776	11
50	9·683284	9·942517	9·740767	10·259233	9·696775	9·938258	9·758517	10·241483	10
51	9·683514	9·942448	9·741066	10·258934	9·696995	9·938185	9·758810	10·241190	9
52	9·683743	9·942378	9·741365	10·258635	9·697215	9·938113	9·759102	10·240898	8
53	9·683972	9·942308	9·741664	10·258336	9·697435	9·938040	9·759395	10·240605	7
54	9·684201	9·942239	9·741962	10·258038	9·697654	9·937967	9·759687	10·240313	6
55	9·684430	9·942169	9·742261	10·257739	9·697874	9·937895	9·759979	10·240021	5
56	9·684658	9·942099	9·742559	10·257441	9·698094	9·937822	9·760272	10·239728	4
57	9·684887	9·942029	9·742858	10·257142	9·698313	9·937749	9·760564	10·239436	3
58	9·685115	9·941959	9·743156	10·256844	9·698532	9·937676	9·760856	10·239144	2
59	9·685343	9·941889	9·743454	10·256546	9·698751	9·937604	9·761148	10·238852	1
60	9·685571	9·941819	9·743752	10·256248	9·698970	9·937531	9·761439	10·238561	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	61 Deg.				60 Deg.				

	34 Deg.				35 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·747562	9·918574	9·828987	10·171013	9·758591	9·913365	9·845227	10·154773	60
1	9·747749	9·918489	9·829260	10·170740	9·758772	9·913276	9·845496	10·154504	59
2	9·747936	9·918404	9·829532	10·170468	9·758952	9·913187	9·845764	10·154236	58
3	9·748123	9·918318	9·829805	10·170195	9·759132	9·913099	9·846033	10·153967	57
4	9·748310	9·918233	9·830077	10·169923	9·759312	9·913010	9·846302	10·153698	56
5	9·748497	9·918147	9·830349	10·169651	9·759492	9·912922	9·846570	10·153430	55
6	9·748683	9·918062	9·830621	10·169379	9·759672	9·912833	9·846839	10·153161	54
7	9·748870	9·917976	9·830893	10·169107	9·759852	9·912744	9·847108	10·152892	53
8	9·749056	9·917891	9·831165	10·168835	9·760031	9·912655	9·847376	10·152624	52
9	9·749243	9·917805	9·831437	10·168563	9·760211	9·912566	9·847644	10·152356	51
10	9·749429	9·917719	9·831709	10·168291	9·760390	9·912477	9·847913	10·152087	50
11	9·749615	9·917634	9·831981	10·168019	9·760569	9·912388	9·848181	10·151819	49
12	9·749801	9·917548	9·832253	10·167747	9·760748	9·912299	9·848449	10·151551	48
13	9·749987	9·917462	9·832525	10·167475	9·760927	9·912210	9·848717	10·151283	47
14	9·750172	9·917376	9·832796	10·167204	9·761106	9·912121	9·848986	10·151014	46
15	9·750358	9·917290	9·833068	10·166932	9·761285	9·912031	9·849254	10·150746	45
16	9·750543	9·917204	9·833339	10·166661	9·761464	9·911942	9·849522	10·150478	44
17	9·750729	9·917118	9·833611	10·166389	9·761642	9·911853	9·849790	10·150210	43
18	9·750914	9·917032	9·833882	10·166118	9·761821	9·911763	9·850057	10·149943	42
19	9·751099	9·916946	9·834154	10·165846	9·761999	9·911674	9·850325	10·149675	41
20	9·751284	9·916859	9·834425	10·165575	9·762177	9·911584	9·850593	10·149407	40
21	9·751469	9·916773	9·834696	10·165304	9·762356	9·911495	9·850861	10·149139	39
22	9·751654	9·916687	9·834967	10·165033	9·762534	9·911405	9·851129	10·148871	38
23	9·751839	9·916600	9·835238	10·164762	9·762712	9·911315	9·851396	10·148604	37
24	9·752023	9·916514	9·835509	10·164491	9·762889	9·911226	9·851664	10·148336	36
25	9·752208	9·916427	9·835780	10·164220	9·763067	9·911136	9·851931	10·148069	35
26	9·752392	9·916341	9·836051	10·163949	9·763245	9·911046	9·852199	10·147801	34
27	9·752576	9·916254	9·836322	10·163678	9·763422	9·910956	9·852466	10·147534	33
28	9·752760	9·916167	9·836593	10·163407	9·763600	9·910866	9·852733	10·147267	32
29	9·752944	9·916081	9·836864	10·163136	9·763777	9·910776	9·853001	10·176999	31
30	9·753128	9·915994	9·837134	10·162866	9·763954	9·910686	9·853268	10·146732	30
31	9·753312	9·915907	9·837405	10·162595	9·764131	9·910596	9·853535	10·146465	29
32	9·753495	9·915820	9·837675	10·162325	9·764308	9·910506	9·853802	10·146198	28
33	9·753679	9·915733	9·837946	10·162054	9·764485	9·910415	9·854069	10·145931	27
34	9·753862	9·915646	9·838210	10·161784	9·764662	9·910325	9·854336	10·145664	26
35	9·754046	9·915559	9·838487	10·161513	9·764838	9·910235	9·854603	10·145397	25
36	9·754229	9·915472	9·838757	10·161243	9·765015	9·910144	9·854870	10·145130	24
37	9·754412	9·915385	9·839027	10·160973	9·765191	9·910054	9·855137	10·144863	23
38	9·754595	9·915297	9·839297	10·160703	9·765367	9·909963	9·855404	10·144596	22
39	9·754778	9·915210	9·839568	10·160432	9·765544	9·909873	9·855671	10·144329	21
40	9·754960	9·915123	9·839838	10·160162	9·765720	9·909782	9·855938	10·144062	20
41	9·755143	9·915035	9·840108	10·159892	9·765896	9·909691	9·856204	10·143796	19
42	9·755326	9·914948	9·840378	10·159622	9·766072	9·909601	9·356471	10·143529	18
43	9·755508	9·914860	9·840648	10·159352	9·766247	9·909510	9·856737	10·143263	17
44	9·755690	9·914773	9·840917	10·159083	9·766423	9·909419	9·857004	10·142996	16
45	9·755872	9·914685	9·841187	10·158813	9·766598	9·909328	9·857270	10·142730	15
46	9·756054	9·914598	9·841457	10·158543	9·766774	9·909237	9·857537	10·142463	14
47	9·756236	9·914510	9·841727	10·158273	9·766949	9·909146	9·857803	10·142197	13
48	9·756418	9·914422	9·841996	10·158004	9·767124	9·909055	9·858069	10·141931	12
49	9·756600	9·914334	9·842266	10·157734	9·767300	9·908964	9·858336	10·141664	11
50	9·756782	9·914246	9·842535	10·157465	9·767475	9·908873	9·858602	10·141398	10
51	9·756963	9·914158	9·842805	10·157195	9·767649	9·908781	9·858868	10·141132	9
52	9·757144	9·914070	9·843074	10·156926	9·767824	9·908690	9·859134	10·140866	8
53	9·757326	9·913982	9·843343	10·156657	9·767999	9·908599	9·859400	10·140600	7
54	9·757507	9·913894	9·843612	10·156388	9·768173	9·908507	9·859666	10·140334	6
55	9·757688	9·913806	9·843882	10·156118	9·768348	9·908416	9·859932	10·140068	5
56	9·757869	9·913718	9·844151	10·155849	9·768522	9·908324	9.860198	10·139802	4
57	9·758050	9·913630	9·844420	10·155580	9·768697	9·908233	9·860464	10·139536	3
58	9·758230	9·913541	9·844689	10·155311	9·768871	9·908141	9·860730	10·139270	2
59	9·758411	9·913453	9·844958	10·155042	9·769045	9·908049	9·860995	10·139005	1
60	9·758591	9·913365	9·845227	10·154773	9·769219	9·907958	9·861261	10·138739	
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	
	55 Deg.				54 Deg.				

	32 Deg.				33 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·724210	9.928420	9·795789	10·204211	9·736109	9·923591	9·812517	10·187483	60
1	9·724412	9·928342	9·796070	10·203930	9·736303	9·923509	9·812794	10·187206	59
2	9·724614	9 928263	9·796351	10·203649	9 736498	9·923427	9·813070	10·186930	58
3	9·724816	9·928183	9·796632	10·203368	9·736692	9·923345	9·813347	10·186653	57
4	9·725017	9·928104	9·796913	10·203087	9·736886	9·923263	9·813623	10·186377	56
5	9·725219	9·928025	9·797194	10·202806	9·737080	9·923181	9·813899	10·186101	55
6	9·725420	9·927946	9·797474	10 202526	9·737274	9·923098	9·814176	10·185824	54
7	9·725622	9·927867	9·797755	10·202245	9·737467	9·923016	9·814452	10·185548	53
8	9·725823	9·927787	9·798036	10·201964	9·737661	9·922933	9·814728	10·185272	52
9	9·726024	9·927708	9·798316	10·201684	9·737855	9·922851	9·815004	10·184996	51
10	9·726225	9·927629	9·798596	10·201404	9 738048	9·922768	9·815280	10·184720	50
11	9·726426	9·927549	9·798877	10·201123	9 738241	9·922686	9·815555	10·184445	49
12	9·726626	9 927470	9·799157	10·200843	9·738434	9·922603	9·815831	10·184169	48
13	9·726827	9·927390	9·799437	10·200563	9·738627	9 922520	9·816107	10·183893	47
14	9·727027	9·927310	9·799717	10·200283	9·738820	9·922438	9 816382	10·183618	46
15	9·727228	9 927231	9·799997	10·200003	9·739013	9·922355	9·816658	10·183342	45
16	9 727428	9 927151	9·800277	10·199723	9·739206	9 922272	9·816933	10·183067	44
17	9·727628	9·927071	9·800557	10·199443	9·739398	9·922189	9·817209	10·182791	43
18	9·727828	9·926991	9·800836	10·199164	9·739590	9·922106	9·817484	10·182516	42
19	9 728027	9·926911	9·801116	10·198884	9·739783	9·922023	9·817759	10·182241	41
20	9·728227	9 926831	9·801396	10·198604	9·739975	9·921940	9·818035	10·181965	40
21	9·728427	9·926751	9·801675	10·198325	9·740167	9·921857	9·818310	10·181690	39
22	9·728626	9·926671	9·801955	10·198045	9·740359	9·921774	9 818585	10·181415	38
23	9·728825	9·926591	9·802234	10·197766	9·740550	9·921691	9·818860	10·181140	37
24	9·729024	9·926511	9·802513	10·197487	9·740742	9·921607	9 819135	10·180865	36
25	9·729223	9·926431	9·802792	10·197208	9·740934	9·921524	9·819410	10·180590	35
26	9·729422	9·926351	9·803072	10·196928	9·741125	9·921441	9·819684	10·180316	34
27	9·729621	9·926270	9·803351	10·196649	9 741316	9·921357	9·819959	10·180041	33
28	9·729820	9·926190	9·803630	10·196370	9·741508	9·921274	9·820234	10·179766	32
29	9·730018	9·926110	9·803909	10·196091	9·741699	9·921190	9·820508	10·179492	31
30	9·730217	9 926029	9·804187	10·195813	9·741889	9·921107	9·820783	10·179217	30
31	9·730415	9·925949	9·804466	10·195534	9·742080	9·921023	9·821057	10·178943	29
32	9·730613	9·925868	9·804745	10·195255	9·742271	9 920939	9·821332	10·178668	28
33	9·730811	9·925788	9·805023	10·194977	9·742462	9·920856	9·821606	10·178394	27
34	9·731009	9·925707	9·805302	10·194698	9·742652	9·920772	9·821880	10·178120	26
35	9·781206	9·925626	9·805580	10·194420	9·742842	9·920688	9·822154	10·177846	25
36	9·731404	9·925545	9·805859	10·194141	9·743033	9·920604	9·822429	10·177571	24
37	9·751602	9 925465	9·806137	10·193863	9·743225	9·920520	9·822703	10·177297	23
38	9·731799	9·925384	9·806415	10·193585	9·743413	9·920436	9·822977	10 177023	22
39	9·731996	9·925303	9 806693	10·193307	9·743602	9·920352	9·823251	10·176749	21
40	9·732193	9·925222	9·806971	10·193029	9·743792	9·920268	9·823524	10·176476	20
41	9·732390	9·925141	9·807249	10.192751	9·743982	9·920184	9·823798	10·176202	19
42	9·732587	9·925060	9·807527	10·192473	9 744171	9 920099	9·824072	10·175928	18
43	9·732784	9·924979	9·807805	10·192195	9·744361	9·920015	9·824345	10·175655	17
44	9·732980	9 924897	9·808083	10·191917	9·744550	9·919931	9·824619	10·175381	16
45	9·733177	9·924816	9·808361	10·191639	9·744739	9·919846	9·824893	10 175107	15
46	9·733373	9 924735	9·808638	10·191362	9·744928	9·919762	9·825166	10·174834	14
47	9·733569	9·924654	9·808916	10·191084	9·745117	9·919677	9·825439	10·174561	13
48	9·733765	9·924572	9 809193	10·190807	9·745306	9·919593	9·825713	10·174287	12
49	9·733961	9·924491	9·809471	10·190529	9·745494	9·919508	9·825986	10 174014	11
50	9·734157	9·924409	9·809748	10·190252	9·745683	9·919424	9·826259	10·173741	10
51	9·734353	9·924328	9·810025	10·189975	9·745871	9·919339	9·826532	10·173468	9
52	9·734549	9·924246	9·810302	10·189698	9·746060	9 919254	9·826805	10·173195	8
53	9·734744	9·924164	9·810580	10·189420	9·746248	9·919169	9·827078	10·172922	7
54	9 734939	9·924083	9·810857	10·189143	9·746436	9·919085	9·827351	10·172649	6
55	9·735135	9·924001	9·811134	10·188866	9·746624	9·919000	9·827624	10·172376	5
56	9·735330	9·923919	9·811410	10·188590	9·746812	9·918915	9·827897	10·172103	4
57	9·735525	9·923837	9·811687	10 188313	9·746999	9·918830	9·828170	10·171830	3
[illegible]	·735719	9·923755	9·811964	10·188036	9·747187	9·918745	9·828442	10·171558	2
[illegible]	735914	9·923673	9·812241	10·187759	6·747374	9·918659	9·828715	10·171285	1
[illegible]	736109	9·923591	9·812517	10·187483	9·747562	9·918574	9·828987	10·171013	0
	sine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	57 Deg.				56 Deg.				

	34 Deg.				35 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·747562	9·918574	9·828987	10·171013	9·758591	9·913365	9·845227	10·154773	60
1	9·747749	9·918489	9·829260	10·170740	9·758772	9·913276	9·845496	10·154504	59
2	9·747936	9·918404	9·829532	10·170468	9·758952	9·913187	9·845764	10·154236	58
3	9·748123	9·918318	9·829805	10·170195	9·759132	9·913099	9·846033	10·153967	57
4	9·748310	9·918233	9·830077	10·169923	9·759312	9·913010	9·846302	10·153698	56
5	9·748497	9·918147	9·830349	10·169651	9·759492	9·912922	9·846570	10·153430	55
6	9·748683	9·918062	9·830621	10·169379	9·759672	9·912833	9·846839	10·153161	54
7	9·748870	9·917976	9·830893	10·169107	9·759852	9·912744	9·847108	10·152892	53
8	9·749056	9·917891	9·831165	10·168835	9·760031	9·912655	9·847376	10·152624	52
9	9·749243	9·917805	9·831437	10·168563	9·760211	9·912566	9·847644	10·152356	51
10	9·749429	9·917719	9·831709	10·168291	9·760390	9·912477	9·847913	10·152087	50
11	9·749615	9·917634	9·831981	10·168019	9·760569	9·912388	9·848181	10·151819	49
12	9·749801	9·917548	9·832253	10·167747	9·760748	9·912299	9·848449	10·151551	48
13	9·749987	9·917462	9·832525	10·167475	9·760927	9·912210	9·848717	10·151283	47
14	9·750172	9·917376	9·832796	10·167204	9·761106	9·912121	9·848986	10·151014	46
15	9·750358	9·917290	9·833068	10·166932	9·761285	9·912031	9·849254	10·150746	45
16	9·750543	9·917204	9·833339	10·166661	9·761464	9·911942	9·849522	10·150478	44
17	9·750729	9·917118	9·833611	10·166389	9·761642	9·911853	9·849790	10·150210	43
18	9·750914	9·917032	9·833882	10·166118	9·761821	9·911763	9·850057	10·149943	42
19	9·751099	9·916946	9·834154	10·165846	9·761999	9·911674	9·850325	10·149675	41
20	9·751284	9·916859	9·834425	10·165575	9·762177	9·911584	9·850593	10·149407	40
21	9·751469	9·916773	9·834696	10·165304	9·762356	9·911495	9·850861	10·149139	39
22	9·751654	9·916687	9·834967	10·165033	9·762534	9·911405	9·851129	10·148871	38
23	9·751839	9·916600	9·835238	10·164762	9·762712	9·911315	9·851396	10·148604	37
24	9·752023	9·916514	9·835509	10·164491	9·762889	9·911226	9·851664	10·148336	36
25	9·752208	9·916427	9·835780	10·164220	9·763067	9·911136	9·851931	10·148069	35
26	9·752392	9·916341	9·836051	10·163949	9·763245	9·911046	9·852199	10·147801	34
27	9·752576	9·916254	9·836322	10·163678	9·763422	9·910956	9·852466	10·147534	33
28	9·752760	9·916167	9·836593	10·163407	9·763600	9·910866	9·852733	10·147267	32
29	9·752944	9·916081	9·836864	10·163136	9·763777	9·910776	9·853001	10·176999	31
30	9·753128	9·915994	9·837134	10·162866	9·763954	9·910686	9·853268	10·146732	30
31	9·753312	9·915907	9·837405	10·162595	9·764131	9·910596	9·853535	10·146465	29
32	9·753495	9·915820	9·837675	10·162325	9·764308	9·910506	9·853802	10·146198	28
33	9·753679	9·915733	9·837946	10·162054	9·764485	9·910415	9·854069	10·145931	27
34	9·753862	9·915646	9·838210	10·161784	9·764662	9·910325	9·854336	10·145664	26
35	9·754046	9·915559	9·838487	10·161513	9·764838	9·910235	9·854603	10·145397	25
36	9·754229	9·915472	9·838757	10·161243	9·765015	9·910144	9·854870	10·145130	24
37	9·754412	9·915385	9·839027	10·160973	9·765191	9·910054	9·855137	10·144863	23
38	9·754595	9·915297	9·839297	10·160703	9·765367	9·909963	9·855404	10·144596	22
39	9·754778	9·915210	9·839568	10·160432	9·765544	9·909873	9·855671	10·144329	21
40	9·754960	9·915123	9·839838	10·160162	9·765720	9·909782	9·855938	10·144062	20
41	9·755143	9·915035	9·840108	10·159892	9·765896	9·909691	9·856204	10·143796	19
42	9·755326	9·914948	9·840378	10·159622	9·766072	9·909601	9·856471	10·143529	18
43	9·755508	9·914860	9·840648	10·159352	9·766247	9·909510	9·856737	10·143263	17
44	9·755690	9·914773	9·840917	10·159083	9·766423	9·909419	9·857004	10·142996	16
45	9·755872	9·914685	9·841187	10·158813	9·766598	9·909328	9·857270	10·142730	15
46	9·756054	9·914598	9·841457	10·158543	9·766774	9·909237	9·857537	10·142463	14
47	9·756236	9·914510	9·841727	10·158273	9·766949	9·909146	9·857803	10·142197	13
48	9·756418	9·914422	9·841996	10·158004	9·767124	9·909055	9·858069	10·141931	12
49	9·756600	9·914334	9·842266	10·157734	9·767300	9·908964	9·858336	10·141664	11
50	9·756782	9·914246	9·842535	10·157465	9·767475	9·908873	9·858602	10·141398	10
51	9·756963	9·914158	9·842805	10·157195	9·767649	9·908781	9·858868	10·141132	9
52	9·757144	9·914070	9·843074	10·156926	9·767824	9·908690	9·859134	10·140866	8
53	9·757326	9·913982	9·843343	10·156657	9·767999	9·908599	9·859400	10·140600	7
54	9·757507	9·913894	9·843612	10·156388	9·768173	9·908507	9·859666	10·140334	6
55	9·757688	9·913806	9·843882	10·156118	9·768348	9·908416	9·859932	10·140068	5
56	9·757869	9·913718	9·844151	10·155849	9·768522	9·908324	9.860198	10·139802	4
57	9·758050	9·913630	9·844420	10·155580	9·768697	9·908233	9·860464	10·139536	3
58	9·758230	9·913541	9·844689	10·155311	9·768871	9·908141	9·860730	10·139270	2
59	9·758411	9·913453	9·844958	10·155042	9·769045	9·908049	9·860995	10·139005	1
60	9·758591	9·913365	9·845227	10·154773	9·769219	9·907958	9·861261	10·138739	
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	
	55 Deg.				54 Deg.				

	36 Deg.				37 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·769219	9·907958	9·861261	10·138739	9·779463	9·902349	9·877114	10·122886	60
1	9·769393	9·907866	9·861527	10·138473	9·779631	9·902253	9·877377	10·122623	59
2	9·769566	9·907774	9·861792	10·138208	9·779798	9·902158	9·877640	10·122360	58
3	9·769740	9·907682	9·862058	10·137942	9·779966	9·902063	9·877903	10·122097	57
4	9·769913	9·907590	9·862323	10·137677	9·780133	9·901967	9·878165	10·121835	56
5	9·770087	9·907498	9·862589	10·137411	9·780300	9·901872	9·878428	10·121572	55
6	9·770260	9·907406	9·862854	10·137146	9·780467	9·901776	9·878691	10·121309	54
7	9·770433	9·907314	9·863119	10·136881	9·780634	9·901681	9·878953	10·121047	53
8	9·770606	9·907222	9·863385	10·136615	9·780801	9·901585	9·879216	10·120784	52
9	9·770779	9·907129	9·863650	10·136350	9·780968	9·901490	9·879478	10·120522	51
10	9·770952	9·907037	9·863915	10·136085	9·781134	9·901394	9·879741	10·120259	50
11	9·771125	9·906945	9·864180	10·135820	9·781301	9·901298	9·880003	10·119997	49
12	9·771298	9·906852	9·864445	10·135555	9·781468	9·901202	9·880265	10·119735	48
13	9·771470	9·906760	9·864710	10·135290	9·781634	9·901106	9·880528	10·119472	47
14	9·771643	9·906667	9·864975	10·135025	9·781800	9·901010	9·880790	10·119210	46
15	9·771815	9·906575	9·865240	10·134760	9·781966	9·900914	9·881052	10·118948	45
16	9·771987	9·906482	9·865505	10·134495	9·782132	9·900818	9·881314	10·118686	44
17	9·772159	9·906389	9·865770	10·134230	9·782298	9·900722	9·881577	10·118423	43
18	9·772331	9·906296	9·866035	10·133965	9·782464	9·900626	9·881839	10·118161	42
19	9·772503	9·906204	9·866300	10·133700	9·782630	9·900529	9·882101	10·117899	41
20	9·772675	9·906111	9·866564	10·133436	9·782796	9·900433	9·882363	10·117637	40
21	9·772847	9·906018	9·866829	10·133171	9·782961	9·900337	9·882625	10·117375	39
22	9·773018	9·905925	9·867094	10·132906	9·783127	9·900240	9·882887	10·117113	38
23	9·773190	9·905832	9·867358	10·132642	9·783292	9·900144	9·883148	10·116852	37
24	9·773361	9·905739	9·867623	10·132377	9·783458	9·900047	9·883410	10·116590	36
25	9·773533	9·905645	9·867887	10·132113	9·783623	9·899951	9·883672	10·116328	35
26	9·773704	9·905552	9·868152	10·131848	9·783788	9·899854	9·883934	10·116066	34
27	9·773875	9·905459	9·868416	10·131584	9·783953	9·899757	9·884196	10·115804	33
28	9·774046	9·905366	9·868680	10·131320	9·784118	9·899660	9·884457	10·115543	32
29	9·774217	9·905272	9·868945	10·131055	9·784282	9·899564	9·884719	10·115281	31
30	9·774388	9·905179	9·869209	10·130791	9·784447	9·899467	9·884980	10·115020	30
31	9·774558	9·905085	9·869473	10·130527	9·784612	9·899370	9·885242	10·114758	29
32	9·774729	9·904992	9·869737	10·130263	9·784776	9·899273	9·885504	10·114496	28
33	9·774899	9·904898	9·870001	10·129999	9·784941	9·899176	9·885765	10·114235	27
34	9·775070	9·904804	9·870265	10·129735	9·785105	9·899078	9·886026	10·113974	26
35	9·775240	9·904711	9·870529	10·129471	9·785269	9·898981	9·886288	10·113712	25
36	9·775410	9·904617	9·870793	10·129207	9·785433	9·898884	9·886549	10·113451	24
37	9·775580	9·904523	9·871057	10·128943	9·785597	9·898787	9·886811	10·113189	23
38	9·775759	9·904429	9·871321	10·128679	9·785761	9·898689	9·887072	10·112928	22
39	9·775920	9·904335	9.871585	10·128415	9·785925	9·898592	9·887333	10·112667	21
40	9·776090	9·904241	9·871849	10·128151	9·786089	9·898494	9·887594	10·112406	20
41	9·776259	9·904147	9·872112	10·127888	9·786252	9·898397	9·887855	10·112145	19
42	9·776429	9·904053	9·872376	10·127624	9·786416	9·898299	9·888116	10·111884	18
43	9·776598	9·903959	9·872640	10·127360	9·786579	9·898202	9·888378	10·111622	17
44	9·776768	9·903864	9·872903	10·127097	9·786742	9·898104	9·888639	10·111361	16
45	9·776937	9·903770	9·873167	10·126833	9·786906	9·898006	9·888900	10·111100	15
46	9·777106	9·903676	9·873430	10·126570	9·787069	9·897908	9·889161	10·110839	14
47	9·777275	9·903581	9·873694	10·126306	9·787232	9·897810	9·889421	10·110579	13
48	9·777444	9·903487	9·873957	10·126043	9·787395	9·897712	9·889682	10·110318	12
49	9·777613	9·903392	9·874220	10·125780	9·787557	9·897614	9·889943	10·110057	11
50	9·777781	9·903398	9·874484	10·125516	9·787720	9·897516	9·890204	10·109796	10
51	9·777950	9·903203	9·874747	10·125253	9·787883	9·897418	9·890465	10·109535	9
52	9·778119	9·903108	9·875010	10·124990	9·788045	9·897320	9·890725	10·109275	8
53	9·778287	9·903014	9·875273	10·124727	9·788208	9·897222	9·890986	10·109014	7
54	9·778455	9·902919	9·875537	10·124463	9·788370	9·897123	9·891247	10·108753	6
55	9·778624	9·902824	9·875800	10·124200	9·788532	9·897025	9·891507	10·108493	5
56	9·778792	9·902729	9·876063	10·123937	9·788694	9·896926	9·891768	10·108232	4
57	9·778960	9·902634	9·876326	10·123674	9·788856	9·896828	9·892028	10·107972	3
58	9·779128	9·902539	9·876589	10·123411	9·789018	9·896729	9·892289	10·107711	2
59	9·779295	9·902444	9·876852	10·123148	9·789180	9·896631	9·892549	10·107451	1
60	9·779463	9·902349	9·877114	10·122886	9·789342	9·896532	9·892810	10·107190	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	53 Deg.				52 Deg.				

′	38 Deg. Sine	Cosine	Tang.	Cotang.	39 Deg. Sine	Cosine	Tang.	Cotang.	
0	9·789342	9·896532	9·892810	10·107190	9·798872	9·890503	9·908369	10·091631	60
1	9·789504	9·896433	9·893070	10·106930	9·799028	9·890400	9·908628	10·091372	59
2	9·789665	9·896335	9·893331	10·106669	9·799184	9·890298	9·908886	10·091114	58
3	9·789827	9·896236	9·893591	10·106409	9·799339	9·890195	9·909144	10·090856	57
4	9·789988	9·896137	9·893851	10·106149	9·799495	9·890093	9·909402	10·090598	56
5	9·790149	9·896038	9·894111	10·105889	9·799651	9·889990	9·909660	10·090340	55
6	9·790310	9·895939	9·894372	10·105628	9·799806	9·889888	9·909918	10·090082	54
7	9·790471	9·895840	9·894632	10·105368	9·799962	9·889785	9·910177	10·089823	53
8	9·790632	9·895741	9·894892	10·105108	9·800117	9·889682	9·910435	10·089565	52
9	9·790793	9·895641	9·895152	10·104848	9·800272	9·889579	9·910693	10·089307	51
10	9·790954	9·895542	9·895412	10·104588	9·800427	9·889477	9·910951	10·089049	50
11	9·791115	9·895443	9·895672	10·104328	9·800582	9·889374	9·911209	10·088791	49
12	9·791275	9·895343	9·895932	10·104068	9·800737	9·889271	9·911467	10·088533	48
13	9·791436	9·895244	9·896192	10·103808	9·800892	9·889168	9·911725	10·088275	47
14	9·791596	9·895145	9·896452	10·103548	9·801047	9·889064	9·911982	10·088018	46
15	9·791757	9·895045	9·896712	10·103288	9·801201	9·888961	9·912240	10·087760	45
16	9·791917	9·894945	9·896971	10·103029	9·801356	9·888858	9·912498	10·087502	44
17	9·792077	9·894846	9·897231	10·102769	9·801511	9·888755	9·912756	10·087244	43
18	9·792237	9·894746	9·897491	10·102509	9·801665	9·888651	9·913014	10·086986	42
19	9·792397	9·894646	9·897751	10·102249	9·801819	9·888548	9·913271	10·086729	41
20	9·792557	9·894546	9·898010	10·101990	9·801973	9·888444	9·913529	10·086471	40
21	9·792716	9·894446	9·898270	10·101730	9·802128	9·888341	9·913787	10·086213	39
22	9·792876	9·894346	9·898530	10·101470	9·802282	9·888237	9·914044	10·085956	38
23	9·793035	9·894246	9·898789	10·101211	9·802436	9·888134	9·914302	10·085698	37
24	9·793195	9·894146	9·899049	10·100951	9·802589	9·888030	9·914560	10·085440	36
25	9·793354	9·894046	9·899308	10·100692	9·802743	9·887926	9·914817	10·085183	35
26	9·793514	9·893946	9·899568	10·100432	9·802897	9·887822	9·915075	10·084925	34
27	9·793673	9·893846	9·899827	10·100173	9·803050	9·887718	9·915332	10·084668	33
28	9·793832	9·893745	9·900087	10·099913	9·803204	9·887614	9·915590	10·084410	32
29	9·793991	9·893645	9·900346	10·099654	9·803357	9·887510	9·915847	10·084153	31
30	9·794150	9·893544	9·900605	10·099395	9·803511	9·887406	9·916104	10·083896	30
31	9·794308	9·893444	9·900864	10·099136	9·803664	9·887302	9·916362	10·083638	29
32	9·794467	9·893343	9·901124	10·098876	9·803817	9·887198	9·916619	10·083381	28
33	9·794626	9·893243	9·901383	10·098617	9·803970	9·887093	9·916877	10·083123	27
34	9·794784	9·893142	9·901642	10·098358	9·804123	9·886989	9·917134	10·082866	26
35	9·794942	9·893041	9·901901	10·098099	9·804276	9·886885	9·917391	10·082609	25
36	9·795101	9·892940	9·902160	10·097840	9·804428	9·886780	9·917648	10·082352	24
37	9·795259	9·892839	9·902420	10·097580	9·804581	9·886676	9·917906	10·082094	23
38	9·795417	9·892739	9·902679	10·097321	9·804734	9·886571	9·918163	10·081837	22
39	9·795575	9·892638	9·902938	10·097062	9·804886	9·886466	9·918420	10·081580	21
40	9·795733	9·892536	9·903197	10·096803	9·805039	9·886362	9·918677	10·081323	20
41	9·795891	9·892432	9·903456	10·096544	9·805191	9·886257	9·918934	10·081066	19
42	9·796049	9·892334	9·903714	10·096286	9·805343	9·886152	9·919191	10·080809	18
43	9·796206	9·892233	9·903973	10·096027	9·805495	9·886047	9·919448	10·080552	17
44	9·796364	9·892132	9·904232	10·095768	9·805647	9·885942	9·919705	10·080295	16
45	9·796521	9·892030	9·904491	10·095509	9·805799	9·885837	9·919962	10·080038	15
46	9·796679	9·891929	9·904750	10·095250	9·805951	9·885732	9·920219	10·079781	14
47	9·796836	9·891827	9·905008	10·094992	9·806103	9·885627	9·920476	10·079524	13
48	9·796993	9·891726	9·905267	10·094733	9·806253	9·885522	9·920733	10·079267	12
49	9·797150	9·891624	9·905526	10·094474	9·806406	9·885416	9·920990	10·079710	11
50	9·797307	9·891523	9·905785	10·094215	9·806557	9·885311	9·921247	10·078753	10
51	9·797464	9·891421	9·906043	10·093957	9·806709	9·885205	9·921503	10·078497	9
52	9·797621	9·891319	9·906302	10·093698	9·806860	9·885100	9·921760	10·078240	8
53	9·797777	9·891217	9·906560	10·093440	9·807111	9·884994	9·922017	10·077983	7
54	9·797934	9·891115	9·906819	10·093181	9·807163	9·884889	9·922274	10·077726	6
55	9·798091	9·891013	9·907077	10·092923	9·807314	9·884783	9·922530	10·077470	5
56	9·798247	9·890911	9·907336	10·092664	9·807465	9·884677	9·922784	10·077216	4
57	9·798403	9·890809	9·907594	10·092406	9·807615	9·884572	9·923044	10·076956	3
58	9·798560	9·890707	9·907853	10·092147	9·807766	9·884466	9·923300	10·076700	2
59	9·798716	9·890605	9·908111	10·091889	9·807917	9·884360	9·923557	10·076443	1
60	9·798872	9·890503	9·908369	10·091631	9·808067	9·884254	9·923814	10·076186	0
	Cosine	Sine	Cotang.	Tang.	Cosine	Sine	Cotan.	Tang.	
	51 Deg.				50 Deg.				

	40 Deg.				41 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·808067	9·884254	9·923814	10·076186	9·816943	9·877780	9·939163	10·060837	60
1	9·808218	9·884148	9·924070	10·075930	9·817088	9·877670	9·939418	10·060582	59
2	9·808368	9·884042	9·924327	10·075673	9·817233	9·877560	9·939763	10·060327	58
3	9·808519	9·883936	9·924583	10·075417	9·817379	9·877450	9·939928	10·060072	57
4	9·808669	9·883829	9·924840	10·075160	9·817524	9·877340	9·940183	10·059817	56
5	9·808819	9·883723	9·925096	10·074904	9·817668	9·877230	9·940439	10·059561	55
6	9·808969	9·883617	9·925352	10·074648	9·817813	9·877120	9·940694	10·059306	54
7	9·809119	9·883510	9·925609	10·074391	9·817958	9·877010	9·940949	10·059051	53
8	9·809269	9·883404	9·925865	10·074135	9·818103	9·876899	9·941204	10·058796	52
9	9·809419	9·883297	9·926122	10·073878	9·818247	9·876789	9·941459	10·058541	51
10	9·809569	9·883191	9·926378	10·073622	9·818392	9·876678	9·941713	10·058287	50
11	9·809718	9·883084	9·926634	10·073366	9·818536	9·876568	9·941968	10·058032	49
12	9·809868	9·882977	9·926890	10·073110	9·818681	9·876457	9·942223	10·057777	48
13	9·810017	9·882871	9·927147	10·072853	9·818825	9·876347	9·942478	10·057522	47
14	9·810167	9·882764	9·927403	10·072597	9·818969	9·876236	9·942733	10·057267	46
15	9·810316	9·882657	9·927659	10·072341	9·819113	9·876125	9·942988	10·057012	45
16	9·810465	9·882550	9·927915	10·072085	9·819257	9·876014	9·943243	10·056757	44
17	9·810614	9·882443	9·928171	10·071829	9·819401	9·875904	9·943498	10·056502	43
18	9·810763	9·882336	9·928427	10·071573	9·819545	9·875793	9·943752	10·056248	42
19	9·810912	9·882229	9·928684	10·071316	9·819689	9·875682	9·944007	10·055993	41
20	9·811061	9·882121	9·928940	10·071060	9·819832	9·875571	9·944262	10·055738	40
21	9·811210	9·882014	9·929196	10·070804	9·819976	9·875459	9·944517	10·055483	39
22	9·811358	9·881907	9·929452	10·070548	9·820120	9·875348	9·944771	10·055229	38
23	9·811507	9·881799	9·929708	10·070292	9·820263	9·875237	9·945026	10·054974	37
24	9·811655	9·881692	9·929964	10·070036	9·820406	9·875126	9·945281	10·054719	36
25	9·811804	9·881554	9·930220	10·069780	9·820550	9·875014	9·945535	10·054465	35
26	9·811952	9·881477	9·930475	10·069525	9·820693	9·874903	9·945790	10·054210	34
27	9·812100	9·881369	9·930731	10·069269	9·820836	9·874791	9·946045	10·053955	33
28	9·812248	9·881261	9·930987	10·069013	9·820979	9·874680	9·946299	10·053701	32
29	9·812396	9·881153	9·931243	10·068757	9·821122	9·874568	9·946554	10·053446	31
30	9·812544	9·881046	9·931499	10·068501	9·821265	9·874456	9·946808	10·053192	30
31	9·812692	9·880938	9·931755	10·068245	9·821407	9·874344	9·947063	10·052937	29
32	9·812840	9·880830	9·932010	10·067990	9·821550	9·874232	9·947318	10·052682	28
33	9·812988	9·880722	9·932266	10·067734	9·821693	9·874121	9·947572	10·052428	27
34	9·813135	9·880613	9·932522	10·067478	9·821835	9·874009	9·947827	10·052173	26
35	9·813283	9·880505	9·932778	10·067222	9·821977	9·873896	9·948081	10·051919	25
36	9·813430	9·880397	9·933033	10·066967	9·822120	9·873784	9·948335	10·051665	24
37	9·813578	9·880289	9·933289	10·066711	9·822262	9·873672	9·948590	10·051410	23
38	9·813725	9·880180	9·933545	10·066455	9·822404	9·873560	9·948844	10·051156	22
39	9·813872	9·880072	9·933800	10·066200	9·822546	9·873448	9·949099	10·050901	21
40	9·814019	9·879963	9·934056	10·065944	9·822688	9·873335	9·949353	10·050647	20
41	9·814166	9·879855	9·934311	10·065689	9·822830	9·873223	9·949608	10·050392	19
42	9·814313	9·879746	9·934567	10·065433	9·822972	9·873110	9·949862	10·050138	18
43	9·814460	9·879637	9·934822	10·065178	9·823114	9·872998	9·950116	10·049884	17
44	9·814607	9·879529	9·935078	10·064922	9·823255	9·872885	9·950371	10·049629	16
45	9·814753	9·879420	9·935333	10·064667	9·823397	9·872772	9·950625	10·049375	15
46	9·814900	9·879311	9·935589	10·064411	9·823539	9·872659	9·950879	10·049121	14
47	9·815046	9·879202	9·935844	10·064156	9·823680	9·872547	9·951133	10·048867	13
48	9·815193	9·879093	9·936100	10·063900	9·823821	9·872434	9·951388	10·048612	12
49	9·815339	9·878984	9·936355	10·063645	9·823963	9·872321	9·951642	10·048358	11
50	9·815485	9·878875	9·936611	10·063389	9·824104	9·872208	9·951896	10·048104	10
51	9·815632	9·878766	9·936866	10·063134	9·824245	9·872095	9·952150	10·047850	9
52	9·815778	9·878656	9·937121	10·062879	9·824386	9·871981	9·952405	10·047595	8
53	9·815924	9·878547	9·937377	10·062623	9·824527	9·871868	9·952659	10·047341	7
54	9·816069	9·878438	9·937632	10·062368	9·824668	9·871755	9·952913	10·047087	6
55	9·816215	9·878328	9·957887	10·062113	9·824808	9·871641	9·953167	10·046833	5
56	9·816361	9·878219	9·938142	10·061858	9·824949	9·871528	9·953421	10·046579	4
57	9·816507	9·878109	9·938398	10·061602	9·825090	9·871414	9·953675	10·046325	3
58	9·816652	9·877999	9·938653	10·061347	9·825230	9·871301	9·953929	10·046071	2
59	9·816798	9·877890	9·938908	10·061092	9·825371	9·871187	9·954183	10·045817	1
60	9·816943	9·877780	9·939163	10·060837	9·825511	9·871073	9·954437	10·045563	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	49 Deg.				48 Deg.				

′	42 Deg.				43 Deg.				
′	Sine	Cosine	Tang.	Cotang.	Sine	Cosine	Tang.	Cotang.	
0	9·825511	9·871073	9·954437	10·045563	9·833783	9·864127	9·969656	10·030344	60
1	9·825651	9·870960	9·954691	10·045309	9·833919	9·864010	9·969909	10·030091	59
2	9·825791	9·870846	9·954946	10·045054	9·834054	9·863892	9·970162	10·029838	58
3	9·825931	9·870732	9·955200	10·044800	9·834189	9·863774	9·970416	10·029584	57
4	9·826071	9·870618	9·955454	10·044546	9·834325	9·863656	9·970669	10·029331	56
5	9·826211	9·870504	9·955708	10·044292	9·834460	9·863538	9·970922	10·029078	55
6	9·826351	9·870390	9·955961	10·044039	9·834595	9·863419	9·971175	10·028825	54
7	9·826491	9·870276	9·956215	10·043785	9·834730	9·863301	9·971429	10·028571	53
8	9·826631	9·870161	9·956469	10·043531	9·834865	9·863183	9·971682	10·028318	52
9	9·826770	9·870047	9·956723	10·043277	9·834999	9·863064	9·971935	10·028065	51
10	9·826910	9·869933	9·956977	10·043023	9·835134	9·862946	9·972188	10·027812	50
11	9·827049	9·869818	9·957231	10·042769	9·835269	9·862827	9·972441	10·027559	49
12	9·827189	9·869704	9·957485	10·042515	9·835403	9·862709	9·972695	10·027305	48
13	9·827328	9·869589	9·957739	10·042261	9·835538	9·862590	9·972948	10·027052	47
14	9·827467	9·869474	9·957993	10·042007	9·835672	9·862471	9·973201	10·026799	46
15	9·827606	9·869360	9·958247	10·041753	9·835807	9·862353	9·973454	10·026546	45
16	9·827745	9·869245	9·958500	10·041500	9·835941	9·862234	9·973707	10·026293	44
17	9·827884	9·869130	9·958754	10·041246	9·836075	9·862115	9·973960	10·026040	43
18	9·828023	9·869015	9·959008	10·040992	9·836209	9·861996	9·974213	10·025787	42
19	9·828162	9·868900	9·959262	10·040738	9·836343	9·861877	9·974466	10·025534	41
20	9·828301	9·868785	9·959516	10·040484	9·836477	9·861758	9·974720	10·025280	40
21	9·828439	9·868670	9·959769	10·040231	9·836611	9·861638	9·974973	10·025027	39
22	9·828578	9·868555	9·960023	10·039977	9·836745	9·861519	9·975226	10·024774	38
23	9·828716	9·868440	9·960277	10·039723	9·836878	9·861400	9·975479	10·024521	37
24	9·828855	9·868324	9·960530	10·039470	9·837012	9·861280	9·975732	10·024268	36
25	9·828993	9·868209	9·960784	10·039216	9·837146	9·861161	9·975985	10·024015	35
26	9·829131	9·868093	9·961038	10·038962	9·837279	9·861041	9·976238	10·023762	34
27	9·829269	9·867978	9·961292	10·038708	9·837412	9·860922	9·976491	10·023509	33
28	9·829407	9·867862	9·961545	10·038455	9·837546	9·860802	9·976744	10·023256	32
29	9·829545	9·867747	9·961799	10·038201	9·837679	9·860682	9·976997	10·023003	31
30	9·829683	9·867631	9·962052	10·037948	9·837812	9·860562	9·977250	10·022750	30
31	9·829821	9·867515	9·962306	10·037694	9·837945	9·860442	9·977503	10·022497	29
32	9·829959	9·867399	9·962560	10·037440	9·838078	9·860322	9·977756	10·022244	28
33	9·830097	9·867283	9·962813	10·037187	9·838211	9·860202	9·978009	10·021991	27
34	9·830234	9·867167	9·963067	10·036933	9·838344	9·860082	9·978262	10·021738	26
35	9·830372	9·867051	9·963320	10·036680	9·838477	9·859962	9·978515	10·021485	25
36	9·830509	9·866935	9·963574	10·036426	9·838610	9·859842	9·978768	10·021232	24
37	9·830646	9·866819	9·963828	10·036172	9·838742	9·859721	9·979021	10·020979	23
38	9·830784	9·866703	9·964081	10·035919	9·838875	9·859601	9·979274	10·020726	22
39	9·830921	9·866586	9·964335	10·035665	9·839007	9·859480	9·979527	10·020473	21
40	9·831058	9·866470	9·964588	10·035412	9·839140	9·859360	9·979780	10·020220	20
41	9·831195	9·866353	9·964842	10·035158	9·839272	9·859239	9·980033	10·019967	19
42	9·831332	9·866237	9·965095	10·034905	9·839404	9·859119	9·980286	10·019714	18
43	9·831469	9·866120	9·965349	10·034651	9·839536	9·858998	9·980538	10·019462	17
44	9·831606	9·866004	9·965602	10·034398	9·839668	9·858877	9·980791	10·019209	16
45	9·831742	9·865887	9·965855	10·034145	9·839800	9·858756	9·981044	10·018956	15
46	9·831879	9·865770	9·966109	10·033891	9·839932	9·858635	9·981297	10·018703	14
47	9·832015	9·865653	9·966362	10·033638	9·840064	9·858514	9·981550	10·018450	13
48	9·832152	9·865536	9·966616	10·033384	9·840196	9·858393	9·981803	10·018197	12
49	9·832288	9·865419	9·966869	10·033131	9·840328	9·858272	9·982056	10·017944	11
50	9·832425	9·865302	9·967123	10·032877	9·840459	9·858151	9·982309	10·017691	10
51	9·832561	9·865185	9·967376	10·032624	9·840591	9·858029	9·982562	10·017438	9
52	9·832697	9·865068	9·967629	10·032371	9·840722	9·857908	9·982814	10·017186	8
53	9·832833	9·864950	9·967883	10·032117	9·840854	9·857786	9·983067	10·016933	7
54	9·832969	9·864833	9·968136	10·031864	9·840985	9·857665	9·983320	10·016680	6
55	9·833105	9·864716	9·968389	10·031611	9·841116	9·857543	9·983573	10·016427	5
56	9·833241	9·864598	9·968643	10·031357	9·841247	9·857422	9·983826	10·016174	4
57	9·833377	9·864481	9·968896	10·031104	9·841378	9·857300	9·984079	10·015921	3
58	9·833512	9·864363	9·969149	10·030851	9·841509	9·857178	9·984332	10·015668	2
59	9·833648	9·864245	9·969403	10·030597	9·841640	9·857056	9·984584	10·015416	1
60	9·833783	9·864127	9·969656	10·030344	9·841771	9·856934	9·984837	10·015163	0
	Cosine	Sine	Cotan.	Tang.	Cosine	Sine	Cotan.	Tang.	′
	47 Deg.				46 Deg.				

44 Deg.					
′	Sine	Cosine	Tang.	Cotang.	′
0	9·841771	9·856934	9·984837	10·015163	60
1	9·841902	9·856812	9·985090	10·014910	59
2	9·842033	9·856690	9·985343	10·014657	58
3	9·842163	9·856568	9·985596	10·014404	57
4	9·842294	9·856446	9·985848	10·014152	56
5	9·842424	9·856323	9·986101	10·013899	55
6	9·842555	9·856201	9·986354	10·013646	54
7	9·842685	9·856078	9·986607	10·013393	53
8	9·842815	9·855956	9·986860	10·013140	52
9	9·842946	9·855833	9·987112	10·012888	51
10	9·843076	9·855711	9·987365	10·012635	50
11	9·843206	9·855588	9·987618	10·012382	49
12	9·843336	9·855465	9·987871	10·012129	48
13	9·843466	9.855342	9·988123	10·011877	47
14	9·843595	9·855219	9·988376	10·011624	46
15	9·843725	9·855096	9·988629	10·011371	45
16	9·843855	9·854973	9·988882	10·011118	44
17	9·843984	9·854850	9·989134	10·010866	43
18	9·844114	9·854727	9·989387	10·010613	42
19	9·844243	9·854603	9·989640	10·010360	41
20	9·844372	9·854480	9·989893	10·010107	40
21	9·844502	9·854356	9·990145	10·009855	39
22	9·844631	9·854233	9·990398	10·009602	38
23	9·844760	9·854109	9·990651	10·009349	37
24	9·844889	9·853986	9·990903	10·009097	36
25	9·845018	9·853862	9·991156	10·008844	35
26	9·845147	9·853738	9·991409	10·008591	34
27	9·845276	9·853614	9·991662	10·008338	33
28	9·845405	9·853490	9·991914	10·008086	32
29	9·845533	9·853366	9·992167	10·007833	31
30	9·845662	9·853242	9·992420	10·007580	30
31	9·845790	9·853118	9·992672	10·007328	29
32	9·845919	9·852994	9·992925	10·007075	28
33	9·846047	9·852869	9·993178	10·006822	27
34	9·846175	9·852745	9·993431	10·006569	26
35	9·846304	9·852620	9·993683	10·006317	25
36	9·846432	9·852496	9·993936	10·006064	24
37	9·846560	9·852371	9·994189	10·005811	23
38	9·846688	9·852247	9·994441	10·005559	22
39	9·846816	9·852122	9·994694	10·005306	21
40	9·846944	9·851997	9·994947	10·005053	20
41	9·847071	9·851872	9·995199	10·004801	19
42	9·847199	9·851747	9·995452	10·004548	18
43	9·847327	9·851622	9·995705	10·004295	17
44	9·847454	9·851497	9·995957	10·004043	16
45	9·847582	9·851372	9·996210	10·003790	15
46	9·847709	9·851246	9·996463	10·003537	14
47	9·847836	9·851121	9·996715	10·003285	13
48	9·847964	9·850996	9·996968	10·003032	12
49	9·848091	9·850870	9·997221	10·002779	11
50	9·848218	9·850745	9·997473	10·002527	10
51	9·848345	9·850619	9·997726	10·002274	9
52	9·848472	9·850493	9·997979	10·002021	8
53	9·848599	9·850368	9·998231	10·001769	7
54	9·848726	9·850242	9·998484	10·001516	6
55	9·848852	9·850116	9·998737	10·001263	5
56	9·848979	9·849990	9·998989	10·001011	4
57	9·849106	9·849864	9·999242	10·000758	3
58	9·849232	9·849738	9·999495	10·000505	2
59	9·849359	9·849611	9·999747	10·000253	1
60	9·849485	9·849485	10·00000	10·000000	0
	Cosine	Sine	Cotan.	Tang.	′
45 Deg.					

END OF VOL. II.